1 Non-archimedean analysis on the extended hyperreal line ${}^*\mathbb{R}_{\mathbf{d}}$ and some transcendence conjectures over field $\mathbb{Q}$ and ${}^*\mathbb{Q}_\omega$.

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Abstract. In this paper possible completion of the Robinson non-archimedean field ${}^*\mathbb{R}$ constructed by Dedekind sections. As interesting example I show how, a few simple ideas from non-archimedean analysis on the pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ gives a short clear nonstandard reconstruction for the Euler's original proof of the Goldbach-Euler theorem. Given an analytic function of one complex variable $f \in \mathbb{Q}[z]$,we investigate the arithmetic nature of the values of f at transcendental points.

Contents

2 List of Notation.

${}^*\mathbb{N}_\infty$.. the set of infinite natural numbers
${}^*\mathbb{R}_\infty$..the set of infinite hyper real numbers
$\mathbf{L}_*= \mathbf{L}\left({}^*\mathbb{R}\right)$..the set of the limited members of ${}^*\mathbb{R}$
$\mathbf{I}_* = \mathbf{I}\left({}^*\mathbb{R}\right)$..................................the set of the infinitesimal members of ${}^*\mathbb{R}$
$\mathbf{halo}(x) = \mu\left(x\right) = x + \mathbf{I}_*$..halo (monad) of $x \in \mathbb{R}$
$\mathbf{st}\left(a\right)$..Robinson standard part of $a \in {}^*\mathbb{R}$
${}^*\mathbb{Z}\left[z_1, ..., z_{\mathbf{n}}\right]$....................internal polynomials over ${}^*\mathbb{Z}$ in $\mathbf{n} \in {}^*\mathbb{N}$ variables

${}^*\mathbb{R}\left[z_1, ..., z_{\mathbf{n}}\right]$....................internal polynomials over ${}^*\mathbb{R}$ in $\mathbf{n} \in {}^*\mathbb{N}$ variables
${}^*\mathbb{C}\left[z_1, ..., z_{\mathbf{n}}\right]$....................internal polynomials over ${}^*\mathbb{C}$ in $\mathbf{n} \in {}^*\mathbb{N}$ variables
${}^*\mathbb{R}_{\mathbf{d}}$..Dedekind completion of the field ${}^*\mathbb{R}$
${}^*\mathbb{R}_{\mathbf{c}}$..Cauchy completion of the field ${}^*\mathbb{R}$
$\varepsilon_{\mathbf{d}} = \sup\left[x | x \in \mu\left(0\right)\right] = \inf\left[x | x \in \mathbb{R}_+\right]$..........................$\mu\left(0\right) \subset {}^*\mathbb{R}_{\mathbf{d}}, \mathbb{R}_+ \subset {}^*\mathbb{R}_{\mathbf{d}}$
$\Delta_{\mathbf{d}} = \sup\left(\mathbb{R}_+\right) = \inf\left({}^*\mathbb{R}_{+\infty}\right)$...
$WST\left(\alpha\right)$.........Wattenberg standard part of $\alpha \in \left(-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}}\right)_{{}^*\mathbb{R}_{\mathbf{d}}}$ (Def.1.3.2.3)

ab.p. (α)....................................absorption part of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ (Def 1.3.3.1.1)
$[\alpha]_\varepsilon$..ε-part of $-\Delta_{\mathbf{d}} < \alpha < \Delta_{\mathbf{d}}$
$\left[\alpha|b^{\#}\right]_\varepsilon$..$\varepsilon$-part of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ for a given $b \in {}^*\mathbb{R}$

"Arthur stopped at the steep descent into the quarry, froze in his steps, straining
to look down and into the distance, extending his long neck. Redrick joined him.
But he did not look where Arthur was looking.
Right at their feet the road into the quarry began, torn up many years ago by the
treads and wheels of heavy vehicles. To the right was a white steep slope,

cracked by the heat; the next slope was half excavated, and among the rocks
and rubble stood a dredge, its lowered bucket jammed impotently against the
side of the road. And,as was to be expected, there was nothing else to be seen
on the road..."

Arkady and Boris Strugatsky

"Roadside Picnic"

3 Introduction.

1.Some transcendence conjectures over field $\mathbb{Q}$.In 1873 French mathematician, Charles Hermite, proved that e is transcendental. Coming as it did 100 years after Euler had established the significance of e, this meant that the issue of transcendence was one mathematicians could not afford to ignore.Within 10 years of Hermite's breakthrough,his techniques had been extended by Lindemann and used to add π to the list of known transcendental numbers. Mathematician then tried to prove that other numbers such as $e + \pi$ and $e \times \pi$ are transcendental too,but these questions were too difficult and so no further examples emerged till today's time. The transcendence of e^πhad been proved in1929 by A.O.Gel'fond.

Conjecture 1. The numbers $e + \pi$ and $e \times \pi$ are irrational.

Conjecture 2. The numbers e and π are algebraically independent.
However, the same question with e^π and π has been answered:
Theorem.(Nesterenko, 1996 [22]) The numbers e^π and π are algebraically independent.

During of XX th century,a typical question: is whether $f(\alpha)$ is a transcendental
number for each algebraic number α has been investigated and answered many
authors.Modern result in the case of entire functions satisfying a linear differential equation provides the strongest results, related with Siegel's E-functions [22],[27].Ref. [22] contains references to the subject before 1998, including Siegel E and G functions.

Theorem.(Siegel C.L.) Suppose that $\lambda \in \mathbb{Q}, \lambda \neq -1, -2, ..., \alpha \neq 0$.

$$\varphi_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}$$

Then $\varphi_\lambda(\alpha)$ is a transcendental number for each algebraic number $\alpha \neq 0$.

Given an analytic function of one complex variable $f(z) \in \mathbb{Q}[z]$,we investigate
the arithmetic nature of the values of $f(z)$ at transcendental points.

Conjecture 3. Is whether $f(\alpha)$ is a irrational number for given transcendental number α.
Conjecture 4. Is whether $f(\alpha)$ is a transcendental number for given transcendental number α.
In particular we investigate the arithmetic nature of the values of classical polylogarithms $Li_s(z)$ at transcendental points.

The classical polylogarithms

$$Li_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s}$$

for $s = 1, 2, ...$ and $|z| \leq 1$ with $(s; z) = (1; 1)$, are ubiquitous. The study of the
arithmetic nature of their special values is a fascinating subject [35] very few is known.Several recent investigations concern the values of these functions at $z = 1$: these are the values at the positive integers of Riemann zeta function

$$\zeta(s) = Li_s(z = 1) = \sum_{n\geq 1} \frac{1}{n^s}$$

One knows that $\zeta(3)$ is irrational [36],and that inInitely many values $\zeta(2n+1)$ of the zeta function at odd integers are irrational.
Conjecture 4. Is whether $Li_s(\alpha)$ is a irrational number for given transcendental number α.
Conjecture 5. Is whether $Li_s(\alpha)$ is a transcendental number for given

transcendental number α.

2.Modern nonstandard analysis and non-archimedean analysis on the extended hyperreal line ${}^*\mathbb{R}_{\mathbf{d}}$.Nonstandard analysis, in its early period of development, shortly after having been established by A. Robinson [1],[4],[5] dealt mainly with nonstandard extensions of some traditional mathematical structures. The system of its foundations, referred to as "model-theoretic foundations" was proposed by Robinson and E. Zakon [12]. Their approach was based on the type-theoretic concept of superstructure $V(S)$ over some set of individuals S and its nonstandard extension (enlargement) ${}^*V(S)$, usually constructed as a (bounded) ultrapower of the "standard" superstructure $V(S)$.They formulated few principles concerning the elementary embedding $V(S) \longmapsto {}^*V(S)$, enabling the use of methods of nonstandard analysis without paying much attention to details of construction of the particular nonstandard extension.

In classical Robinsonian nonstandard analysis we usualy deal only with completely internal objects which can defined by internal set theory **IST** introduced by E.Nelson [11]. It is known that **IST** is a conservative extension of ZFC. In **IST** all the classical infinite sets, e.g., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, acquire new, nonstandard elements (like *"infinite" natural numbers* or *"infinitesimal" reals*). At the same time, the families ${}^\sigma\mathbb{N} = \{x \in \mathbb{N} : \mathbf{st}(x)\}$ or ${}^\sigma\mathbb{R} = \{x \in \mathbb{R} : \mathbf{st}(x)\}$ of all standard,i.e., "true," natural numbers or reals, respectively, are not sets in **IST** at all. Thus, for a traditional mathematician inclined to ascribe to mathematical objects a certain kind of objective existence or reality, accepting **IST** would mean confessing that everybody has lived in confusion, mistakenly having regarded as, e.g., the set $\mathbb{N}$ just its tiny part ${}^\sigma\mathbb{N}$ (which is not even a set) and overlooked the rest. Edvard Nelson and Karel Hrbček have improved this lack by introducing several "nonstandard" set theories dealing with standard, internal and *external sets* [13]. Note that in contrast with early period of development of the nonstandard analysis in latest period many mathematicians dealing with external and internal set simultaneously,for example see [14],[15],[16],[17].

Many properties of the standard reals $x \in \mathbb{R}$ suitably reinterpreted, can be transfered to the internal hyperreal number system. For example, we have seen that ${}^*\mathbb{R}$,like $\mathbb{R}$,is a totally ordered field. Also, jast $\mathbb{R}$ contain the natural number $\mathbb{N}$ as a discrete subset with its own characteristic properties,${}^*\mathbb{R}$ contains the hypernaturals ${}^*\mathbb{N}$ as the corresponding discrete subset with analogous properties.For example, the standard archimedean property $\forall x_{x\in\mathbb{R}} \forall y_{y\in\mathbb{R}} \exists n_{n\in\mathbb{N}} [(|x| < |y|) \dashrightarrow n\,|x| \geq |y|]$ is preserved in non-archimedean field ${}^*\mathbb{R}$ in respect hypernaturals ${}^*\mathbb{N}$,i.e. the next property is satisfied $\forall x_{x\in\mathbb{R}} \forall y_{y\in\mathbb{R}} \exists n_{n\in\mathbb{N}} [(|x| < |y|) \dashrightarrow n\,|x| \geq |y|]$. However, there are many fundamental properties of $\mathbb{R}$ do not transfered to ${}^*\mathbb{R}$.

I. This is the case one of the fundamental *supremum property* of the standard totally ordered field $\mathbb{R}$.It is easy to see that it apper bound property does not necesarily holds by considering, for example, the (external) set $\mathbb{R}$ itself which we ragard as canonically imbedded into hyperreals ${}^*\mathbb{R}$. This is a non-empty set which is bounded above (by any of the infinite member in ${}^*\mathbb{R}$) but does not have a least apper bound in ${}^*\mathbb{R}$. However by using transfer one obtain the next statement [18] : **Weak supremum property for** ${}^*\mathbb{R}$

Every non-empty *internal* subset $A \subsetneqq {}^*\mathbb{R}$ which has an apper bound in ${}^*\mathbb{R}$ has

a least apper bound in ${}^*\mathbb{R}$.

This is a problem, because any advanced variant of the analysis on the field ${}^*\mathbb{R}$ is needed more strong fundamental supremum property. At first sight one can improve this lack by using corresponding external constructions which known as Dedekind sections and Dedekind completion (see section **I.3.**).We denote corresponding Dedekind completion by symbol ${}^*\mathbb{R}_{\mathbf{d}}$. It is clear that ${}^*\mathbb{R}_{\mathbf{d}}$ is completely external object. But unfortunately ${}^*\mathbb{R}_{\mathbf{d}}$ is not iven a non-archimedean ring but non-archimedean *pseudo-ring* only. However this lack does not make greater difficulties because non-archimedean pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ contains non-archimedean subfield $\Re_{\mathbf{c}} \subset {}^*\mathbb{R}_{\mathbf{d}}$ such that $\Re_{\mathbf{c}} \approx {}^*\mathbb{R}_{\mathbf{c}}$.Here ${}^*\mathbb{R}_{\mathbf{c}}$ this is a Cauchy completion of the non-archimedean field ${}^*\mathbb{R}$ (see section **I.4.**).

II. This is the case two of the fundamental *Peano's induction property*

$$\forall B\,[[(1 \in B) \wedge \forall x\,(x \in B \implies x+1 \in B)] \implies B = \mathbb{N}] \tag{1}$$

does not necesarily holds for arbitrary subset $B \subset {}^*\mathbb{N}$. Therefore (1) is true for ${}^*\mathbb{N}$ when interpreted in ${}^*\mathbb{N}$ i.e.,

$$\forall^{\mathbf{int}} B\,[[(1 \in B) \wedge \forall x\,(x \in B \implies x+1 \in B)] \implies B = {}^*\mathbb{N}]$$

true for ${}^*\mathbb{N}$ provided that we read "$\forall B$" as "for each internal subset B of ${}^*\mathbb{N}$", i.e. as $\forall^{\mathbf{int}} B$. In general the importance of internal versus external entities rests on the fact that each statement that is true for $\mathbb{R}$ is true for ${}^*\mathbb{R}$ provided its quantifiers are restricted to the internal entities (subset) of ${}^*\mathbb{R}$ only [18].This is a problem, because any advanced variant of the analysis on the field ${}^*\mathbb{R}$ is needed more strong induction property than property (2).In this paper I have improved this lack by using external construction two different types for operation of exteral summation: $Ext-\sum_{n\in\mathbb{N}} q_n$, $\#Ext-\sum_{n\in\mathbb{N}} q_n^{\#}$ and two different types for operation of exteral multiplication: $Ext-\prod_{n\in\mathbb{N}} q_n, \#Ext-\prod_{n\in\mathbb{N}} q_n^{\#}$ for arbitrary countable sequences such as $q_n : \mathbb{N} \to \mathbb{R}$ and $q_n^{\#} : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$.

As interesting example I show how, this external constructions from non-archimedean analysis on the pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ gives a short and clear nonstandard reconstruction for the Euler's original proof of the Goldbach-Euler theorem.

4 I.The classical hyperreals numbers.

5 I.1.1.The construction non-archimedean field ${}^*\mathbb{R}$.

Let $\Re$ denote the ring of real valued sequences with the usual pointwise operations.If x is a real number we let s_x denote the constant sequence,$\mathbf{s}_x = x$ for all n. The function sending x to $\mathbf{s}_x$ is a one-to-one ring homomorphism,providing

an embedding of $\mathbb{R}$ into $\Re$. In the following, wherever it is not too confusing we will not distinguish between $x \in \mathbb{R}$ and the constant function $\mathbf{s}_x$,leaving the reader to derive intent from context. The ring $\Re$ has additive identity 0 and multiplicative identity 1. $\Re$ is not a field because if r is any sequence having 0 in its range it can have no multiplicative inverse. There are lots of zero divisors in $\Re$.

We need several definitions now. Generally, for any set S, $\mathbf{P}(S)$ denotes the set of all subsets of S. It is called the power set of S. Also, a subset of $\mathbb{N}$ will be called cofinite if it contains all but finitely many members of $\mathbb{N}$. The symbol $\varnothing$ denotes the empty set. A partition of a set S is a decomposition of S into a union of sets, any pair of which have no elements in common.

Definition.1.1.1. An ultrafilter $\mathbf{H}$ over $\mathbb{N}$ is a family of sets for which:

(i) $\varnothing \notin \mathbf{H} \subset \mathbf{P}(\mathbb{N}), \mathbb{N} \in \mathbf{H}$.

(ii) Any intersection of finitely many members of $\mathbf{H}$ is in $\mathbf{H}$.

(iii) $A \subset \mathbb{N}, B \in \mathbf{H} \Rightarrow A \cup B \in \mathbf{H}$.

(iv) If $V_1, ..., V_n$ is any finite partition of $\mathbb{N}$ then $\mathbf{H}$ contains exactly one of the V_i.

If, further,

(v) $\mathbf{H}$ contains every cofinite subset of $\mathbb{N}$.

the ultrafilter is called *free.*

If an ultrafilter on $\mathbb{N}$ contains a finite set then it contains a one-point set, and is

nothing more than the family of all subsets of $\mathbb{N}$ containing that point. So if an

ultrafilter is not free it must be of this type, and is called a *principal ultrafilter.*

The existence of a free ultrafilter containing any given infinite subset of $\mathbb{N}$ is

implied by the Axiom of Choice.

Remark 1.1.1. Suppose that $x \in X$. An ultrafilter denoted $\mathbf{prin}_X(x) \subseteq X$

consisting of all subsets $S \subseteq X$ which contain x, and called the *principal*

ultrafilter generated by x.

Proposition 1.1.1. If an ultrafilter $\mathbf{F}$ on X contains a finite set $S \subseteq X$, then $\mathbf{F}$ is

principal.

Proof: It is enough to show $\mathbf{F}$ contains $\{x\}$ for some $x \in S$. If not, then $\mathbf{F}$

contains the complement $X\setminus\{x\}$ for every $x \in S$, and therefore also the finite

intersection $\mathbf{F} \ni \bigcap\limits_{x\in S} X\setminus\{x\} = X\setminus S$, which contradicts the fact that $S \in F$.

It follows that nonprincipal ultrafilters can exist only on infinite sets X, and that

every cofinite subset of X (complement of a finite set) belongs to such an

ultrafilter.

Remark 1.1.2. Our construction below depends on the use of a free-not a principal-ultrafilter.

We are going to be using conditions on sequences and sets to define subsets of $\mathbb{N}$. We introduce a convenient shorthand for the usual "set builder" notation. If P is a property that can be true or false for natural numbers we use $[[P]]$ to denote $\{n \in \mathbb{N} | P(n) \text{ is true }\}$. This notation will only be employed during a discussion to decide if the set of natural numbers defined by P is in $\mathbf{H}$, or not. For example, if s, t is a pair of sequences in $\Re$ we define three sets of integers For example, if s, t is a pair of sequences in S we define three sets of integers

$$[[s < t]], \quad [[s = t]], \ [[s > t]]. \tag{1.1}$$

Since these three sets partition $\mathbb{N}$, exactly one of them is in $\mathbf{H}$, and we

declare $s \equiv t$ when $[[s = t]] \in \mathbf{H}$.

Lemma 1.1.1. $\equiv$ is an equivalence relation on $\Re$. We denote the equivalence

class of any sequence s under this relation by $[s]$. Define for each $r \in \Re$ the

sequence $\tilde{r}$ by

$$\tilde{r} = \left\{ \begin{array}{c} 0 \ \ \textbf{iff} \ r_n = 0 \\ r_n^{-1} \ \textbf{iff} \ r_n \neq 0 \end{array} \right\}. \tag{1.2}$$

Lemma 1.1.2. (**a**) There is at most one constant sequence in any class $[r]$.
(**b**) $[0]$ is an ideal in $\Re$ so $\Re/[0]$ is a commutative ring with identity $[1]$.
(**c**) Consequently $[r] = r + [0] = \{r + t | t \in [0]\}$ for all $r \in \Re$.
(**d**) If $[r] \neq [0]$ then $[\tilde{r}] \cdot [r] = [1]$. So $[r]^{-1} = [\tilde{r}]$.

From Lemma 1.1.2., we conclude that ${}^*\mathbb{R}$, defined to be $\Re/[0]$, is a field

containing an embedded image of as a subfield. $[0]$ is a maximal ideal in $\Re$.

Definition.1.1.2. This quotient ring is called the field ${}^*\mathbb{R}$ of **classical hyperreal**

numbers.

We declare $[s] < [t]$ provided $[[s < t]] \in \mathbf{H}$.

Recall that any field with a linear order $<$ is called an ordered field provided
(i) $x + y > 0$ whenever $x, y > 0$
(ii) $x \cdot y > 0$ whenever $x, y > 0$
(iii) $x + z > y + z$ whenever $x > y$

Theorem 1.1.3. (**a**) The relation given above is a linear order on ${}^*\mathbb{R}$, and makes

${}^*\mathbb{R}$ into an ordered field. As with any ordered field, we define $|x|$ for $x \in {}^*\mathbb{R}$ to

be x or $-x$, whichever is nonnegative.

(b) If x, y are real then $x \leq y$ if and only if $[x] \leq [y]$. So the ring morphism of $\mathbb{R}$ into ${}^*\mathbb{R}$ is also an order isomorphism onto its image in ${}^*\mathbb{R}$.

Because of this last theorem and the essential uniqueness of the real numbers it is common to identify the embedded image of $\mathbb{R}$ in ${}^*\mathbb{R}$ with $\mathbb{R}$ itself. Though obviously circular, one does something similar when identifying $\mathbb{Q}$ with its isomorphic image in ${}^*\mathbb{R}$, and $\mathbb{N}$ itself with the corresponding subset of $\mathbb{Q}$. This kind of notational simplification usually does not cause problems.

Now we get to the ideas that prompted the construction. Define the sequence r by $r_n = (n+1)^{-1}$. For every positive integer k, $[[r < k^{-1}]] \in \mathbf{H}$.So $0 < [r] < 1/k$. We have found a positive hyperreal smaller than (the embedded image of) any real number. This is our first nontrivial infinitesimal number. The sequence $\tilde{r}$ is given by $\tilde{r}_n = n+1$.So $[r]^{-1} = [\tilde{r}] > k$ for every positive integer k.$[r]^{-1}$ is a hyperreal larger than any real number.

6 I.1.2.The brief nonstandard vocabulary.

Definition.1.1.2.1. We call a member $x \in {}^*\mathbb{R}$ $\mathbb{R}$-*limited* if there are members $a, b \in \mathbb{R}$ with $a < x < b$.

We will use $\mathbf{L}_*= \mathbf{L}\,({}^*\mathbb{R})$ to indicate the limited members of ${}^*\mathbb{R}$. x is called $\mathbb{R}$-*unlimited* if it not $\mathbb{R}$-limited.

These terms are preferred to "finite" and "infinite,"
which are reserved for concepts related to cardinality.

Definition.1.1.2.2. If $x, y \in {}^*\mathbb{R}$ and $x < y$ we use ${}^*[x, y]$ to denote $\{t \in {}^*\mathbb{R} | x \leq t \leq y\}$.

This set is called a *closed hyperinterval.* Open and half-open hyperintervals are defined and denoted similarly.

Definition.1.1.2.3. A set $S \subset {}^*\mathbb{R}$ is called *hyperbounded* if there are members
x, y of ${}^*\mathbb{R}$ for which S is a subset of the hyperinterval ${}^*[x, y]$.

Abusing standard vocabulary for ordered sets, S is called bounded if x and y
can be chosen to be limited members of ${}^*\mathbb{R}$ x and y could, in fact, be chosen to be real if S is bounded.

The vocabulary of bounded or hyperbounded above and below can be used.

Definition.1.1.2.3. We call a member $x \in {}^*\mathbb{R}$ infinitesimal if $|x| < a$ for every
positive $a \in \mathbb{R}$. We write $x \approx 0$ iff x is infinitesimal.

The only real infinitesimal is obviously 0.

We will use $\mathbf{I}_* = \mathbf{I}\,({}^*\mathbb{R})$ to indicate the infinitesimal members of ${}^*\mathbb{R}$.

Definition.1.1.2.4. A member $x \in {}^*\mathbb{R}$ is called *appreciable* if it is limited but not
infinitesimal.

Definition.1.1.2.5. Hyperreals x and y are said to have *appreciable separation* if

$|x - y|$ is appreciable.

We will be working with various subsets S of ${}^*\mathbb{R}$ and adopt the following

convention: $S_\infty = S \backslash \mathbf{L}_* = \{x \in S | x \notin \mathbf{L}_*\}$. These are the unlimited members of

S,if any.

Definition.1.1.2.6. (**a**) We say two hyperreals x, y are infinitesimally close or

have infinitesimal separation if $|x - y| \in \mathbf{I}_*$.

We use the notation $x \approx y$ to indicate that x and y are infinitesimally close.

(**b**) They have limited separation if $|x - y| \in \mathbf{L}_*$.

(**c**) Otherwise they are said to have unlimited separation.

We define the halo of x by $\mathbf{halo}(x) = x + \mathbf{I}_*$. There can be at most one real number in any halo. Whenever $\mathbf{halo}(x) \cap \mathbb{R}$ is nonempty we define the shadow

of x, denoted $\mathbf{shad}(x)$, to be that unique real number.

The galaxy of x is defined to be $\mathbf{gal}(x) = x + \mathbf{L}_*$. $\mathbf{gal}(x)$ is the set of hyperreal numbers a limited distance away from x. So if x is limited $\mathbf{gal}(x) = \mathbf{L}_*$.

If n is any fixed positive integer we define ${}^*\mathbb{R}^n$ to be the set of equivalence

classes of sequences in $\mathbb{R}^n$ under the equivalence relation $x \equiv y$ exactly when $[[x = y]] \in \mathbf{H}$.

Definition.1.1.2.7. We call ${}^*\mathbb{N}$ the set of *classical hypernatural or A. Robinson's*

hypernatural numbers, ${}^*\mathbb{N}_\infty$ the set of *classical infinite hypernatural or*

A. Robinson's infinite hypernatural numbers, ${}^*\mathbb{R}_\infty$ the set of *classical infinite*

hyperreal or A. Robinson's infinite hyperreal numbers, ${}^*\mathbb{Z}$ the set of *classical hyperintegers or A. Robinson's hyperintegers*, and ${}^*\mathbb{Q}$ the set of *classical*

hyperrational numbers or A. Robinson's hyperrational numbers.

Theorem 1.1.2.1. ${}^*\mathbb{R}$ is not Dedekind complete.

(hint: $\mathbb{N}$ is bounded above by the member $[\mathbf{t}] \in {}^*\mathbb{N}_\infty$, where $\mathbf{t}$ is the sequence given by $t_n = n$ for all $n \in \mathbb{N}$. But $\mathbb{N}$ can have no least upper bound: if $n \leq c$ for all $n \in \mathbb{N}$ then $n \leq c - 1$ for all $n \in \mathbb{N}$.

As another example consider $\mathbf{I}_*$. This set is (very) bounded, but has no least

upper bound.)

Theorem 1.1.2.2. For every $r \in {}^*\mathbb{R}$ there is unique $n \in {}^*\mathbb{N}$ with $n \leq r < n + 1$.

7 I.2.The higher orders of hyper-method.Second order transfer principle.

8 I.2.1.What are the higher orders of hyper-method?

Usual nonstandard analysis essentially consists only of two fundamental tools:
the (**first order**) **star-map** $*_1 \triangleq *$ and the (**first order**) **transfer principle**.

In most applications, a third fundamental tool is also considered, namely the
saturation property.

Definition.1.2.1.1. Any universe $\mathbf{U}$ is a nonempty collection of *"standard*
mathematical objects" that is closed under subsets, i.e. $a \subsetneqq A \in \mathbf{U} \implies a \in \mathbf{U}$
and closed under the basic mathematical operations.Precisely, whenever
$A, B \in \mathbf{U}$, we require that also the union $A \cup B$, the intersection $A \cap B$,the
set-difference $A \backslash B$ the ordered pair $\{A, B\}$,the Cartesian product $A \times B$, the
powerset $P(A) = \{a | a \subsetneqq A\}$,the function-set $B^A = \{f \mid f : A \to B\}$ all belong
to $\mathbf{U}$.A universe $\mathbf{U}$ is also assumed to contain (copies of) all
sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \in \mathbf{U}$, and to be transitive, i.e. members of
members of $\mathbf{U}$ belong to $\mathbf{U}$ or in formulae: $a \in A \in \mathbf{U} \implies a \in \mathbf{U}$.

The notion of "*standard* mathematical object" includes all objects used in the
ordinary practice of mathematics, namely: numbers, sets, functions, relations,
ordered tuples, Cartesian products, etc. It is well-known that all these notions
can be defined as sets and formalized in the foundational framework
of Zermelo-Fraenkel axiomatic set theory **ZFC**.

From standard assumption: $Con\,(\mathbf{ZFC})$ and Gödel's completeness theorem
one obtain that **ZFC** has a model M. A model M of set theory is called standard
if the element relation $\in_M$of the model M is the actual element relation $\in$
restricted to the model M, i.e.$\in_M \triangleq \in|_M$. A model is called transitive when it is
standard and the base class is a transitive class of sets. A model of set theory
is often assumed to be transitive unless it is explicitly stated that it is

non-standard. Inner models are transitive,transitive models are standard, and standard models are well-founded.

The assumption that there exists a standard model of **ZFC** (in a given universe) is stronger than the assumption that there exists a model. In fact, if there is a standard model, then there is a smallest standard model called the minimal model contained in all standard models. The minimal model contains no standard model (as it is minimal) but (assuming the consistency of **ZFC**) it contains some model of **ZFC** by the Gödel completeness theorem. This model is necessarily not well founded otherwise its Mostowski collapse would be a standard model. (It is not well founded as a relation in the universe, though it satisfies the axiom of foundation so is "internally" well founded. Being well founded is not an absolute property[2].) In particular in the minimal model there is a model of **ZFC** but there is no standard model of **ZFC**.

By the theorem of Löwenheim-Skolem, we can choose transitive models M_ω of **ZFC** of countable cardinality.

Remark 1.2.1.1. In **ZFC**, an ordered pair $\{a, b\}$ is defined as the Kuratowski pair $\{\{a\}, \{a, b\}\}$;an n-tuple is inductively defined by $\{a_1, ..., a_n, a_{n+1}\} = \{\{a_1, ..., a_n\}, a_{n+1}\}$; an n-place relation R on A is identified with the set $R \subseteq A^n$ of n-tuples that satisfy it; a function $f : A \to B$ is identified with its graph $[\{a, b\} \in A \times B | b = f(a)]$.
As for numbers, complex numbers $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ are defined as ordered pairs of real numbers, and the real numbers $\mathbb{R}$ are defined as equivalence classes of suitable sets of rational numbers namely, Dedekind cuts or Cauchy sequences.
The rational numbers $\mathbb{Q}$ are a suitable quotient $\mathbb{Z} \times \mathbb{Z}/_{\approx}$, and the integers $\mathbb{Z}$ are in turn a suitable quotient $\mathbb{N} \times \mathbb{N}/_{\approx}$. The natural numbers of **ZFC** are defined as the set of von Neumann naturals: $0 = \emptyset$ and $n + 1 = \{n\}$ (so that each natural number $\{n = 0, 1, ..., n - 1\}$ is identified with the set of its predecessors.)

Each countable model M_ω of **ZFC** contains countable model $\mathbb{R}_\omega$ of the real numbers $\mathbb{R}$.Every element $x \in \mathbb{R}_\omega$ defines a Dedekind cut:

$\mathbb{Q}_x \triangleq \{q \in \mathbb{Q} | q \leq x\} \cup \{q \in \mathbb{Q} | q > x\}$ .We therefore get a order preserving map

$$f_p : \mathbb{R}_\omega \to \mathbb{R}$$

and which respects addition and multiplication.

We address the question what is the possible range of f_p?

Proposition 1.2.1.1. Choose an arbitrary subset $\Theta \subset \mathbb{R}$.Then there is a model $\mathbb{R}(\Theta)$ such that $f_p[\mathbb{R}(\Theta)] \supset \Theta$.Moreover, the cardinality of $\mathbb{R}(\Theta)$ can be chosen to coincide with Θ, if Θ is infinite.

Proof. Choose $\Theta \subset \mathbb{R}$.For each $\alpha \in \Theta$ choose
$q_1(\alpha) < q_2(\alpha) < ... < p_1(\alpha) < p_2(\alpha)$ with $\lim_{n\to\infty} q_n(\alpha) = \lim_{n\to\infty} p_n(\alpha) = \alpha$.

We add to the axioms of $\mathbb{Q}$ the following axioms:

$$\forall \alpha (\alpha \in \Theta) \exists e_\alpha \forall k (k \in \mathbb{N}) [q_k(\alpha) < e_\alpha$$

Again $\mathbb{Q}$ is a model for each finite subset of these axioms,so that the compactness theorem implies the existence of $\mathbb{R}(\Theta)$ as required,where the cardinality of $\mathbb{R}(\Theta)$ can be chosen to be the cardinality of the set of axioms, i.e. of Θ, if Θ is infinite. Note that by construction $f_p(e_\alpha) = \alpha$.

Remark 1.2.1.2. It follows by the theorem of Löwenheim-Skolem,that for each countable subset $\Theta_\omega \subset \mathbb{R}$ we can find a countable model $\mathbb{R}_\omega = \mathbb{R}(\Theta_\omega)$
of $\mathbb{R}$ such that the image of f_p contains this subset.Note,on the other hand, that the image will only be countable, so that the different models $\mathbb{R}(\Theta_\omega)$ will have very different ranges.

Hyper-Tool # 1: FIRST ORDER STAR-MAP.

Definition.1.2.1.2. The **first order** star-map is a function $*_1 : \mathbf{U} \to \mathbf{V}_1$ between two universes that associates to each object $A \in \mathbf{U}$ its **first order** hyper-extension (or **first order** non-standard extension) ${}^{*_1}A \in \mathbf{V}_1$. It is also assumed that ${}^{*_1}n = n$ for all natural numbers $n \in \mathbb{N}$, and that the properness condition ${}^{*_1}\mathbb{N} \neq \mathbb{N}$ holds.

Remark 1.2.1.1. It is customary to call standard any object $A \in \mathbf{U}$ in the domain of the first order star-map $*_1$, and first order nonstandard any object $B \in \mathbf{V}_1$ in the codomain. The adjective standard is also often used in the literature for first order hyper-extensions ${}^{*_1}A \in \mathbf{V}_1$.

Hyper-Tool # 2: SECOND ORDER STAR-MAP.

Definition.1.2.1.3. The **second order** star-map is a function $*_2 : \mathbf{V}_1 \to \mathbf{V}_2$
between two universes that associates to each object $A \in \mathbf{V}_1$ its **second order** hyper-extension (or **second order** non-standard extension) ${}^{*_2}A \in \mathbf{V}_2$.

It is also assumed that ${}^{*_2}N = N$ for all hyper natural numbers $N \in {}^{*_1}\mathbb{N}$, and

that the properness condition ${}^{*_2}\mathbb{N} \neq {}^{*_1}\mathbb{N}$ holds.

Hyper-Tool # 3: FIRST ORDER TRANSFER PRINCIPLE.

Definition.1.2.1.4. Let $P(a_1, ..., a_n)$ be a property of the standard objects $a_1, ..., a_n \in \mathbf{U}$ expressed as an "elementary sentence". Then $P(a_1, ..., a_n)$ is true if and only if corresponding sentence ${}^{*_1}P\left(c_1, ..., c_n\right)$ is true about the corresponding hyper-extensions ${}^{*_1}a_1, ..., {}^{*_1}a_n \in \mathbf{V}_1$. That is:

$$P(a_1, ..., a_n) \iff {}^{*_1}P\left({}^{*_1}a_1, ..., {}^{*_1}a_n\right).$$

In particular $P(a_1, ..., a_n)$ is true if and only if the same sentence $P\left(c_1, ..., c_n\right)$ is true about the corresponding hyper-extensions ${}^{*_1}a_1, ..., {}^{*_1}a_n \in \mathbf{V}_1$. That is:

$P(a_1, ..., a_n) \iff P\left({}^{*_1}a_1, ..., {}^{*_1}a_n\right).$

Hyper-Tool # 4: SECOND ORDER TRANSFER PRINCIPLE.

Definition.1.2.1.5. Let ${}^{*_1}P\left({}^{*_1}a_1, ..., {}^{*_1}a_n\right)$ be a property of the **first order** non-standard objects ${}^{*_1}a_1, ..., {}^{*_1}a_n \in \mathbf{V}_1$ expressed as an "elementary sentence".

9 I.2.2.The higher orders of hyper-method by using countable universes.

Definition.1.2.2.1. Any countable universe $\mathbf{U}_\omega$ is a nonempty countable collection of *"standard mathematical objects"* that is closed under subsets, i.e. $a \subsetneqq A \in \mathbf{U} \implies a \in \mathbf{U}$ and closed under the basic mathematical operations.

Precisely, whenever $A, B \in \mathbf{U}$, we require that also the union $A \cup B$, the intersection $A \cap B$,the set-difference $A \backslash B$ the ordered pair $\{A, B\}$,the Cartesian product $A \times B$, the powerset $P(A) = \{a | a \subsetneqq A\}$,the function-set $B^A = \{f \mid f : A \to B\}$ all belong to $\mathbf{U}_\omega$.A countable universe $\mathbf{U}_\omega$ is also assumed to contain (copies of) all sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q} \in \mathbf{U}_\omega, \mathbb{R}_\omega, \mathbb{C}_\omega \in \mathbf{U}_\omega$, and to be transitive, i.e. members of members of $\mathbf{U}_\omega$ belong to $\mathbf{U}_\omega$ or in formulae: $a \in A \in \mathbf{U}_\omega \implies a \in \mathbf{U}_\omega$.

Remark 1.2.2.1. In any countable model M_ω of **ZFC**,an ordered pair $\{a, b\}$ is defined as the Kuratowski pair

$\{\{a\},\{a,b\}\}$;an n-tuple is inductively defined by $\{a_1,...,a_n,a_{n+1}\} =$
$\{\{a_1,...,a_n\},a_{n+1}\}$; an n-place relation R on A is identified with the countable set
$R \subseteq A^n$ of n-tuples that satisfy it; a function $f : A \to B$ is identified with its graph
$[\{a,b\} \in A \times B | b = f(a)]$.

As for numbers, complex numbers $\mathbb{C}_\omega = \mathbb{R}_\omega \times \mathbb{R}_\omega$ are defined as ordered pairs of
real numbers, and the real numbers $\mathbb{R}_\omega$ are defined as countable set of
countable equivalence classes of suitable sets of rational numbers namely,
Dedekind cuts or Cauchy sequences.

The rational numbers $\mathbb{Q}$ are a suitable quotient $\mathbb{Z} \times \mathbb{Z}/_{\approx}$, and the integers $\mathbb{Z}$ are
in turn a suitable quotient $\mathbb{N} \times \mathbb{N}/_{\approx}$. The natural numbers of **ZFC** are defined as
the set of von Neumann naturals: $0 = \emptyset$ and $n+1 = \{n\}$ (so that each natural
number $\{n = 0,1,...,n-1\}$ is identified with the set of its predecessors.)

10 I.2.3.Divisibility of hyperintegers.

Definition.1.2.3.1. If n and d are hypernaturals,i.e. $n,d \in {}^*\mathbb{N}$ or hyperintegers, i.e. $n,d \in {}^*\mathbb{Z}$ and $d \neq 0$,then n is divisible by d provided $n = d \cdot k$ for some hyperinteger k.Alternatively, we say:

1.n is a multiple of d,
2.d is a factor of n,
3.d is a divisor of n,
4.d divides n (denoted with $d \mid n$).

Theorem 1.2.3.1. Transitivity of Divisibility.
For all $a,b,c \in {}^*\mathbb{Z}$, if $a|b$ and $b|c$, then $a|c$.

Theorem 1.2.3.2. Every positive hyperinteger greater than 1
is divisible by a hyperprime number.

Definition.1.2.3.2. Given any integer $n > 1$, the
standard factored form of $n \in {}^*\mathbb{N}$ is an expression of $n = {}^*\prod_{k=1}^{m} p_k^{e_k}$,
where m is a positive hyperinteger, $p_1,p_2,...,p_m$ are
hyperprime numbers with $p_1 < p_2 < ... < p_m$ and
$e_1,e_2,...,e_m$ are positive hyperintegers.

Theorem 1.2.3.3. Given any hyperinteger $n > 1$, there exist
positive hyperinteger m,hyperprime numbers $p_1,p_2,...,p_m$

and positive hyperintegers $e_1,e_2,...,e_m$ with $n = {}^*\prod_{k=1}^{m} p_k^{e_k}$.

Theorem 1.2.3.1. (**i**) Every pair of elements $m, n \in {}^*\mathbb{N}$ has a highest common factor $d = s \times m + t \times n$ for some $s, t \in {}^*\mathbb{Z}$.
(**ii**) For every pair of elements $a, d \in {}^*\mathbb{N}$ dividend a and divisor d, with $d \neq 0$ there exist unique integers q and r such that $a = q \times d + r$ and $0 \leq r < |d|$.

Definition.1.2.3.2. Suppose that $a = q \times d + r$ and $0 \leq r < |d|$ .We call d the
quotient and r the remainder.

Redrick, squinting his swollen eyes against the blinding light, silently watched him go. He was cool and calm, he knew what was about to happen, and he knew that he would not watch,but it was still all right to watch, and he did, feeling nothing in particular,except that deep inside a little worm started wriggling around and twisting its sharp head in his gut.

Arkady and Boris Strugatsky

”Roadside Picnic”

11 I.3.The construction non-archimedean pseudo-ring${}^*\mathbb{R}_{\mathbf{d}}$.

12 I.3.1.Generalized pseudo-ring Dedekind hyperreals ${}^*\mathbb{R}_{\mathbf{d}}$.

From **Theorem 1.2.1.** above we knov that: ${}^*\mathbb{R}$ **is not Dedekind complete**. For example, $\mu(0)$ and $\mathbb{R}$ are bounded subsets of ${}^*\mathbb{R}$ which have no suprema or infima in ${}^*\mathbb{R}$.
Possible standard completion of the field ${}^*\mathbb{R}$ can be constructed by Dedekind sections [23],[24]. In [24] Wattenberg constructed the Dedekind completion of a nonstandard model of the real numbers and applied the construction to obtain certain kinds of special measures on the set of integers. Thus was established that the Dedekind completion ${}^*\mathbb{R}_{\mathbf{d}}$ of the field ${}^*\mathbb{R}$ is a structure of interest not for its own sake only and we establish further importent
applications here. Importent concept was introduce Gonshor [23] is that of the **absorption number** of an element $\mathbf{a} \in {}^*\mathbb{R}_{\mathbf{d}}$ which, roughly speaking, measures the degree to which the cancellation law $\mathbf{a} + b = \mathbf{a} + c \implies b = c$ fails for $\mathbf{a}$.
More general construction well known from topoi theory [10].

Definition 1.3.1.1. A *Dedekind hyperreal* $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ is a pair

$(U,V) \in \mathbf{P}\left({}^{*}\mathbb{Q}\right) \times \mathbf{P}\left({}^{*}\mathbb{Q}\right)$ satisfying the next conditions:
1. $\exists x \exists y \left(x \in U \wedge y \in V\right)$.
2. $U \cap V = \varnothing$.
3. $\forall x \left(x \in U \iff \exists y \left(y \in V \wedge x < y\right)\right)$.
4. $\forall x \left(x \in V \iff \exists y \left(y \in V \wedge y < x\right)\right)$.
5. $\forall x \forall y \left(x < y \implies x \in U \vee y \in V\right)$.

Remark. The monad of $\alpha \in {}^{*}\mathbb{R}$, the set $\{x \in {}^{*}\mathbb{R} |\ x \approx \alpha\}$ is denoted: $\mu(\alpha)$.
Monad $\mu(0)$ is denoted: $\mathbf{I}_{*}$.Supremum of $\mathbf{I}_{*}$ is denoted: $\varepsilon_{\mathbf{d}}$.
Let A be a subset of ${}^{*}\mathbb{R}$ is bounded or hyperbounded above then
$\sup(A)$ exists in ${}^{*}\mathbb{R}_{\mathbf{d}}$.

Example. (**i**) $\Delta_{\mathbf{d}} = \sup\left(\mathbb{R}_{+}\right) \in {}^{*}\mathbb{R}_{\mathbf{d}} \setminus {}^{*}\mathbb{R}$,(**ii**) $\varepsilon_{\mathbf{d}} = \sup\left(\ \mathbf{I}_{*}\right) \in {}^{*}\mathbb{R}_{\mathbf{d}} \setminus {}^{*}\mathbb{R}$.
Remark. Anfortunately the set ${}^{*}\mathbb{R}_{\mathbf{d}}$ inherits some but by **no means all** of the algebraic structure on ${}^{*}\mathbb{R}$.For example,${}^{*}\mathbb{R}_{\mathbf{d}}$ is not a group with respect to addition since if $x +_{{}^{*}\mathbb{R}_{\mathbf{d}}} y$ denotes the addition in ${}^{*}\mathbb{R}_{\mathbf{d}}$ then:

$$\varepsilon_{\mathbf{d}} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} 0_{{}^{*}\mathbb{R}_{\mathbf{d}}} = \varepsilon_{\mathbf{d}}$$

Thus ${}^{*}\mathbb{R}_{\mathbf{d}}$ is not iven a ring but pseudo-ring only. Thus, one must proceed somewhat cautiously. In this section more details than is customary will be included in proofs because some standard properties which at first glance appear clear often at second glance reveal themselves to be false in ${}^{*}\mathbb{R}_{\mathbf{d}}$.

We shall briefly remind a way Dedekind's constructions of a pseudo-field

${}^{*}\mathbb{R}_{\mathbf{d}}$.

Definition 1.3.1.2.a.(General) Suppose $\preceq$ is a total ordering on X. Then $\{A,B\}$ is said to be a Dedekind cut of $\langle X,\preceq\rangle$,if and only if:
1. A and B are nonempty subsets of X.
2. $A \cup B = X$.
3. For each x in A and each y in $B, x \preceq y$.
A is the *left-hand* part of the cut $\{A,B\}$ and B is the *riht-hand* part

of the cut $\{A,B\}$. We denote the cut as $x = A|B$ or simple $x = A$.
The cut $x = A|B$ is less than or equal to the cut $y = C|D$ if $A \subsetneqq C$.
We write $x \preceq y$ if x is less than or equal to y and we write $x \prec y$ if $x \preceq y$ and $x \neq y$.
Definition 1.3.2.b. $c \in X$ is said to be a cut element of $\{A,B\}$ if and only if either:
(**i**) c is in A and $x \preceq c \preceq y$ for each x in A and each y in B, or
(**ii**) c is in B and $x \preceq c \preceq y$ for each x in A and each y in B.
Definition 1.3.2.c.$\langle X,\preceq\rangle$ is said to be *Dedekind complete* if and only if

each Dedekind cut of $\langle X,\preceq\rangle$,has a cut element.
Example. The following theorem is well-known.

Theorem. $\langle \mathbb{R}, \leqslant \rangle$ is Dedekind complete, and for each Dedekind cut $\{A, B\}$,of $\langle \mathbb{R}, \leqslant \rangle$ if r and s are cut elements of $\{A, B\}$, then $r = s$.
Making a semantic leap, we now answer the question "what is a *Dedekind hyperreal number* ?"

Definition 1.3.2.d. A **Dedekind hyperreal number** is a cut in ${}^*\mathbb{Q}$.
${}^*\mathbb{R}_{\mathbf{d}}$ is the class of all Dedekind hyperreal numbers $x = A|B$ $(x = A)$.
We will show that in a natural way ${}^*\mathbb{R}_{\mathbf{d}}$ is a **complete ordered generalized pseudo-ring** containing ${}^*\mathbb{R}$.
Before spelling out what this means, here are some examples of cuts.

(**i**)
$A|B = \{r \in {}^*\mathbb{Q} \mid r < 1\} | \{r \in {}^*\mathbb{Q} \mid r \geq 1\}$.

(**ii**)
$A|B = \left\{r \in {}^*\mathbb{Q} \mid (r \leq 0) \vee \left(r^2 < 2\right)\right\} \left| \left\{r \in {}^*\mathbb{Q} \mid (r > 0) \wedge \left(r^2 \geq 2\right)\right\}\right.$.

(**iii**)
$A|B = \{r \in {}^*\mathbb{Q} \mid r < \omega\} | \{r \in {}^*\mathbb{Q} \mid r \geq \omega\}$,where $\omega \in {}^*\mathbb{Q}_+ \backslash \mathbb{Q}_+$.

(**iv**)
$A|B =$
$\{r \in {}^*\mathbb{Q} \mid (r \leq 0) \vee (r \in \mathbf{I}_*) \vee (r \in \mathbb{Q}_+)\}|$
$\{r \in {}^*\mathbb{Q} \mid (r > 0) \wedge \ (r \in {}^*\mathbb{Q}_+ \backslash (\mathbb{Q}_+ \cup \mathbf{I}_*))\}$.

(**v**)
$A|B =$
$\{r \in {}^*\mathbb{Q} \mid (r \leq 0) \vee (r \in \mathbf{I}_*)\} | \{r \in {}^*\mathbb{Q} \mid (r > 0) \wedge \ (r \in {}^*\mathbb{Q}_+ \backslash \mathbf{I}_*)\}$.

Remark. It is convenient to say that $A|B \in {}^*\mathbb{R}_{\mathbf{d}}$ is a **rational** (**hyperrational**)

cut in ${}^*\mathbb{Q}$ if it is like the cut in examples (**i**),(**iii**): fore some fixed rational (hyperrational) number $c \in {}^*\mathbb{Q}, A$ is the set of all hyperrational r such that $r < c$ while B is the rest of ${}^*\mathbb{Q}$.

The B-set of a rational (hyperrational) cut contains a smollest $c \in {}^*\mathbb{Q}$, and conversaly if $A|B$ is a cut in ${}^*\mathbb{Q}$ and B contains a smollest element c then $A|B$ is a rational or hyperrational cut at c.We write $\check{c}$ for the rational hyperrational cut at c.This lets us think of ${}^*\mathbb{Q} \subset {}^*\mathbb{R}_{\mathbf{d}}$ by identifying c with $\check{c}$.

Remark. It is convenient to say that:

(**1**) $A|B \in {}^*\mathbb{R}_{\mathbf{d}}$ is an **standard cut** in ${}^*\mathbb{Q}$ if it is like the cut in examples (**i**)-(**ii**):
fore some cut $A'|B' \in \mathbb{R}$ the next equality is satisfied:$A|B = {}^*(A')\,|{}^*(B')$,i.e. A-set of a cut is an standard set.

(**2**) $A|B \in {}^*\mathbb{R}_{\mathbf{d}}$ is an **internal cut** or **nonstandard cut** in ${}^*\mathbb{Q}$ if it is like the cut in example (**iii**), i.e. A-set of a cut is an *internal nonstandard set.*

(**3**) $A|B \in {}^*\mathbb{R}_{\mathbf{d}}$ is an **external cut** in ${}^*\mathbb{Q}$ if it is like the cut in examples (**iv**)-(**v**),

i.e. A-set of a cut is an *external set.*

There is an order relation $(\cdot \leq \cdot)$ on cuts that fairly cries out for attention.

Definition 1.3.2.e. The cut $x = A|B$ is less than or equal to the cut $y = C|D$

if $A \subseteqq C$.

We write $x \leq y$ if x is less than or equal to y and we write $x < y$ if

$x \leq y$ and $x \neq y$.If $x = A|B$ is less than $y = C|D$ then $A \subset C$ and $A \neq C$,so

there is some $c_0 \in C \backslash A$.Sinse the A-set of a cut contains no largest element,

there is also a $c_1 \in C$ with $c_0 < c_1$.All the hyperrational numbers c with

$c_0 \leq c \leq c_1$ belong to $C \backslash A$.

Remark. The property distinguishing ${}^*\mathbb{R}_{\mathbf{d}}$ from ${}^*\mathbb{Q}$ and from ${}^*\mathbb{R}$ and which is

the bottom of every significant theorem about ${}^*\mathbb{R}_{\mathbf{d}}$ involves upper bounds and

least upper bounds or equivalently,lower bounds and gretest lower bounds.

Definition 1.3.2.f. $M \in {}^*\mathbb{R}_{\mathbf{d}}$ is an **upper bound** for a set $S \subset {}^*\mathbb{R}_{\mathbf{d}}$ if each $s \in S$

satisfies $s \leq M$. We also say that the set S is **bounded above** by M iff

$M \in \mathbf{L}\left({}^*\mathbb{R}\right)$ We also say that the set S is **hyperbounded above** iff $M \notin \mathbf{L}\left({}^*\mathbb{R}\right)$,

i.e.$|M| \in {}^*\mathbb{R}_{+} \backslash \mathbb{R}_{+}$.

Definition 1.3.2.g. An upper bound for S that is less than all other upper

bound for S is a **least upper bound** for S.

The concept of a *pseudo-ring* originally was introduced by

E. M.Patterson [21].Briefly,Patterson's pseudo-ring is an algebraic

system consisting of an additive abelian group $\mathbf{A}$, a distinguished

subgroup $\mathbf{A}^*$ of $\mathbf{A}$,and a multiplication operation $\mathbf{A}^* \times \mathbf{A} \to \mathbf{A}$ under

which $\mathbf{A}^*$ is a ring and $\mathbf{A}$ a left $\mathbf{A}^*$-module.For convenience, we denote

the pseudo-ring by $\Re = (A^*, A)$.

Definition 1.3.1.2.h.Generalized pseudo-ring (m-pseudo-ring) is an

algebraic system consisting of an abelian semigroup $\mathbf{A_s}$ (or abelian monoid

$\mathbf{A_m}$),a distinguished subgroup $\mathbf{A_s^*}$ of $\mathbf{A_s}$ (or a distinguished subgroup $\mathbf{A_m^*}$

of $\mathbf{A_m}$)**,**and a multiplication operation $\mathbf{A_s^*} \times \mathbf{A_s} \to \mathbf{A_s}$ ($\mathbf{A_m^*} \times \mathbf{A_m} \to \mathbf{A_m}$) under

which $\mathbf{A_s^*}$ ($\mathbf{A_m^*}$) is a ring and $\mathbf{A_s}$ ($\mathbf{A_m}$) a left $\mathbf{A_s^*}$-module ($\mathbf{A_m^*}$-module).

For convenience,we denote the generalized pseudo-ring by $\Re_{\mathbf{s}} = (A_{\mathbf{s}}^*, A_{\mathbf{s}})$.

Pseudo-field is an algebraic system consisting of an abelian semigroup $\mathbf{A_s}$**,**

a distinguished subgroups $\mathbf{A_s^*} \subsetneqq \mathbf{A_s^\#}$ of $\mathbf{A_s}$ and a multiplication operations

$\mathbf{A_s^*} \times \mathbf{A_s} \to \mathbf{A_s}$and $\mathbf{A_s} \times \mathbf{A_s^*} \to \mathbf{A_s}$ under which $\mathbf{A_s^*}$ is a ring,$\mathbf{A_s^\#}$ is a field and $\mathbf{A_s}$

is a vector spase over field $\mathbf{A_s^\#}$.

Definition 1.3.1.2. A Dedekind cut α in ${}^*\mathbb{Q}$ is a subset $\alpha \subset {}^*\mathbb{Q}$ of the

hyperrational numbers ${}^*\mathbb{Q}$ that satisfies these properties:
1. α is not empty.
2. $\beta = {}^*\mathbb{Q}\backslash\alpha$ is not empty.
3. α contains no greatest element
4. For $x, y \in {}^*\mathbb{Q}$, if $x \in \alpha$ and $y < x$, then $y \in \alpha$ as well.
Definition 1.3.1.3. A Dedekind hyperreal number $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ is a Dedekind cut α in ${}^*\mathbb{Q}$. We denote the set of all Dedekind hyperreal numbers by ${}^*\mathbb{R}_{\mathbf{d}}$ and we order them by set-theoretic inclusion, that is to say, for any $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}$, $\alpha < \beta$ if and only if $\alpha \subsetneqq \beta$ where the inclusion is strict. We further define $\alpha = \beta$ as real numbers if and are equal as sets. As usual, we write $\alpha \leqslant \beta$ if $\alpha < \beta$ or $\alpha = \beta$.
Definition 1.3.1.4. A hyperreal number α is said to be *Dedekind hyperirrational* if ${}^*\mathbb{Q}\backslash\alpha$ contains no least element.
Theorem 1.3.1.1. Every nonempty subset $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ of Dedekind hyperreal numbers that is bounded (hyperbounded) above has a least upper bound.
Proof. Let A be a nonempty set of hyperreal numbers, such that for every $\alpha \in A$ we have that $\alpha \leqslant \gamma$ for some real number $\gamma \in {}^*\mathbb{R}_{\mathbf{d}}$.
Now define the set $\sup A = \bigcup_{\alpha \in A} \alpha$. We must show that this set is a Dedekind hyperreal number. This amounts to checking the four conditions
of a Dedekind cut. $\sup A$ is clearly not empty, for it is the nonempty union of nonempty sets. Because γ is a Dedekind hyperreal number, there is some hyperrational $x \in {}^*\mathbb{Q}$ that is not in γ. Since every $\alpha \in A$ is a subset of γ, x is not
in any α, so $x \notin \sup A$ either. Thus, ${}^*\mathbb{Q}\backslash \sup A$ is nonempty. If $\sup A$ had a greatest
element $g \in {}^*\mathbb{Q}$, then $g \in \alpha$ for some $\alpha \in A$. Then g would be a greatest element of α, but α is a Dedekind hyperreal number, so by contrapositive law, $\sup A$ has no greatest element. Lastly, if $x \in {}^*\mathbb{Q}$ and $x \in \sup A$, then $x \in \alpha$ for
some α, so given any $y \in {}^*\mathbb{Q}$, $y < x$ because α is a Dedekind hyperreal number
$y \in \alpha$ whence $y \in \sup A$.Thus $\sup A$, is a Dedekind hyperreal number.

Trivially,$\sup A$ is an upper bound of A, for every $\alpha \in A$, $\alpha \subseteqq \sup A$. It now suffices
to prove that $\sup A \leqslant \gamma$,because was an arbitrary upper bound. But this is easy,
because every $x \in \sup A, x \in {}^*\mathbb{Q}$ is an element of α for some $\alpha \in A$, so because $\alpha \subseteq \gamma, x \in \gamma$. Thus, $\sup A$ is the least upper bound of A.
Definition 1.3.1.5. Given two Dedekind hyperreal numbers α and β we define:
1.The additive identity (zero cut) $0_{{}^*\mathbb{R}_{\mathbf{d}}}$, denoted 0,is

$0_{^*\mathbb{R}_\mathbf{d}} \triangleq \{x \in {}^*\mathbb{Q}|\ x < 0\}$.

2. The multiplicative identity $1_{^*\mathbb{R}_\mathbf{d}}$, denoted 1,is

$1_{^*\mathbb{R}_\mathbf{d}} \triangleq \{x \in {}^*\mathbb{Q}|\ x <_{^*\mathbb{Q}} 1_{^*\mathbb{Q}}\}$.

3. Addition $\alpha +_{^*\mathbb{R}_\mathbf{d}} \beta$ of α and β denoted $\alpha + \beta$ is

$\alpha + \beta \triangleq \{x + y|\ x \in \alpha, y \in \beta\}$.

It is easy to see that $\alpha +_{^*\mathbb{R}_\mathbf{d}} 0_{^*\mathbb{R}_\mathbf{d}} = 0_{^*\mathbb{R}_\mathbf{d}}$ for all $\alpha \in {}^*\mathbb{R}_\mathbf{d}$.

It is easy to see that $\alpha +_{^*\mathbb{R}_\mathbf{d}} \beta$ is a cut in ${}^*\mathbb{Q}$ and $\alpha +_{^*\mathbb{R}_\mathbf{d}} \beta = \beta +_{^*\mathbb{R}_\mathbf{d}} \alpha$.

Another fundamental property of cut addition is associativity:

$(\alpha +_{^*\mathbb{R}_\mathbf{d}} \beta) +_{^*\mathbb{R}_\mathbf{d}} \gamma = \alpha +_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma)$.

This follows from the corresponding property of ${}^*\mathbb{Q}$.

4. The opposite $-_{^*\mathbb{R}_\mathbf{d}}\alpha$ of α, denoted $-\alpha$, is

$-\alpha \triangleq \{x \in {}^*\mathbb{Q}|\ -x \notin \alpha, -x \text{ is not the least element of } {}^*\mathbb{Q}\backslash\alpha\}$.

5.Remark. We also say that the opposite $-\alpha$ of α is the **additive inverse**

of α denoted $\div\alpha$ iff the next equality is satisfied: $\alpha + (\div\alpha) = 0$.

6.Remark. It is easy to see that for all internal cut $\alpha^{\mathbf{Int}}$ the opposite $-\alpha^{\mathbf{Int}}$ is the additive inverse of $\alpha^{\mathbf{Int}}$,i.e. $\alpha^{\mathbf{Int}} + \left(\div\alpha^{\mathbf{Int}}\right) = 0$.

7.Example. (External cut X without additive inverse $\div X$) For any $x, y \in {}^*\mathbb{R}$ we

denote: $x \lll \infty \triangleq \exists r\left[(r \in \mathbb{R}) \wedge (x < {}^*r)\right], y \approx -\infty \triangleq \forall r\left[r \in \mathbb{R} \implies x < {}^*r\right]$.

Let us consider two Dedekind hyperreal numbers X and Y defined as:

$X = \{x|\,(x \in {}^*\mathbb{R}) \wedge (x \lll \infty)\}, Z = \{x|z < 0\}$.

It is easy to see that is no exist cut Y such that: $X + Y = Z$.

Proof. Suppose that cut Y such that: $X + Y = Z$ exist. It is easy to check that

$\forall y\left[y \in Y \implies y \approx -\infty\right]$.Suppose that $y \in Y$,then $\forall x\left[x \in X \implies x + y \in Z\right]$,

i.e.$\forall\,(x \lll \infty)\left[x + y < 0\right]$.Hence $\forall\,(x \lll \infty)\left[y < -x\right]$,i.e. $y \approx -\infty$.It is easy to check

that $Z \subsetneqq X + Y$.If $x \in X$ and $y \in Y$ then $x \lll \infty$ and $y \approx -\infty$,hence $x + y \neq -1$,i.e.

$-1 \notin X + Y$.Thus $Z \subsetneqq X + Y$.This is a contradiction.

8. We say that the cut α is positive if $0 < \alpha$ or negative if $\alpha < 0$.

The absolute value of α,denoted $|\alpha|$,is $|\alpha| \triangleq \alpha$,if $\alpha \geq 0$ and $|\alpha| \triangleq -\alpha$,if $\alpha \leq 0$

9. If $\alpha, \beta > 0$ then multiplication $\alpha \times_{^*\mathbb{R}_\mathbf{d}} \beta$ of α and β denoted $\alpha \times \beta$ is

$\alpha \times \beta \triangleq \{z \in {}^*\mathbb{Q}|\ z = x \times y \text{ for some } x \in \alpha, y \in \beta \text{ with } x, y > 0\}$.

In general, $\alpha \times \beta = 0$ if $\alpha = \mathbf{0}$ or $\beta = 0$,

$\alpha \times \beta \triangleq |\alpha| \times |\beta|$ if $\alpha > 0, \beta > 0$ or $\alpha < 0, \beta < 0$,

$\alpha \times \beta \triangleq -(|\alpha| \cdot |\beta|)$ if $\alpha > 0, \beta < 0$,or $\alpha < 0, \beta > 0$.

10. The cut order enjois on ${}^*\mathbb{R}_\mathbf{d}$ the standard additional properties of:

(i) **transitivity:** $\alpha \leq \beta \leq \gamma \implies \alpha \leq \gamma$.

(ii) **trichotomy:** eizer $\alpha < \beta, \beta < \alpha$ or $\alpha = \beta$ but only one of the three things is true.

(iii) **translation:** $\alpha \leq \beta \implies \alpha +_{^*\mathbb{R}_\mathbf{d}} \gamma \leq \beta +_{^*\mathbb{R}_\mathbf{d}} \gamma$.

11. By definition above, this is what we mean when we say that ${}^*\mathbb{R}_\mathbf{d}$ is an

ordered pseudo-ring or **ordered pseudo-field.**
Remark. We embed ${}^*\mathbb{R}$ in ${}^*\mathbb{R}_{\mathbf{d}}$ in the standard way [24].If $\alpha \in {}^*\mathbb{R}$ the corresponding element, $\alpha^{\#}$, of ${}^*\mathbb{R}_{\mathbf{d}}$ is

$$\alpha^{\#} \triangleq \{x \in {}^*\mathbb{R} | x <_{{}^*\mathbb{R}} \alpha\}$$

Lemma 1.3.1.1.[24].
(**i**) Addition $(\circ +_{{}^*\mathbb{R}_{\mathbf{d}}} \circ)$ is commutative and associative in${}^*\mathbb{R}_{\mathbf{d}}$.
(**ii**) $\forall \alpha \in {}^*\mathbb{R} : \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} 0_{{}^*\mathbb{R}_{\mathbf{d}}} = \alpha$.
(**iii**) $\forall \alpha, \beta \in {}^*\mathbb{R} : \alpha^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta^{\#} = (\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{\#}$.
Proof. (**i**) Is clear from definitions.
(**ii**) Suppose $a \in \alpha$. Since a has no greatest element $\exists b \left[(b > a) \wedge (b \in \alpha)\right]$. Thus $a - b \in 0_{{}^*\mathbb{R}_{\mathbf{d}}}$ and $a = (a - b) + b \in 0_{{}^*\mathbb{R}_{\mathbf{d}}} + \alpha$.
(**iii**) (a) $\alpha^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta^{\#} \subseteqq (\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{\#}$ is clear since:
$(x < \alpha) \wedge (y < \beta) \implies x + y < \alpha + \beta$.
(b) Suppose $x < \alpha + \beta$. Thus $\alpha - \dfrac{(\alpha+\beta)-x}{2} < \alpha$ and $\beta - \dfrac{(\alpha+\beta)-x}{2} < \beta$.
So one obtain $x = \left[\left(\alpha - \dfrac{(\alpha+\beta)-x}{2}\right) + \left(\beta - \dfrac{(\alpha+\beta)-x}{2}\right)\right] \in \alpha^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta^{\#}$,
$(\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{\#} \subseteqq \alpha^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta^{\#}$.
Notice, here again something is lost going from ${}^*\mathbb{R}$ to ${}^*\mathbb{R}_{\mathbf{d}}$ since $a < \beta$ does not imply $\alpha + \alpha < \beta + \alpha$ since $0 < \varepsilon_{\mathbf{d}}$ but $0 + \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}} + \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}}$.
Lemma 1.3.1.2.[24].
(i) $\leq_{{}^*\mathbb{R}_{\mathbf{d}}}$a linear ordering on ${}^*\mathbb{R}_{\mathbf{d}}$,which extends the usual ordering on ${}^*\mathbb{R}$.
(ii) $(\alpha \leq \alpha') \wedge (\beta \leq \beta') \implies \alpha + \beta \leq \alpha' + \beta'$.
(iii) $(\alpha < \alpha') \wedge (\beta < \beta') \implies \alpha + \beta < \alpha' + \beta'$.
(iv) ${}^*\mathbb{R}$ **is dense in** ${}^*\mathbb{R}_{\mathbf{d}}$.That is if $\alpha < \beta$ in ${}^*\mathbb{R}_{\mathbf{d}}$ there is an $a \in {}^*\mathbb{R}$ then $\alpha < a^{\#} < \beta$.

Lemma 1.3.1.3.[24].
(i) If $\alpha \in {}^*\mathbb{R}$ then $-_{{}^*\mathbb{R}_{\mathbf{d}}} (\alpha^{\#}) = (-_{{}^*\mathbb{R}}\alpha)^{\#}$.
(ii) $-_{{}^*\mathbb{R}_{\mathbf{d}}} (-_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha) = \alpha$.
(iii) $\alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \iff -_{{}^*\mathbb{R}_{\mathbf{d}}}\beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} -_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha$.
(iv) $(-_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha) +_{{}^*\mathbb{R}_{\mathbf{d}}} \left(-_{{}^*\mathbb{R}_{\mathbf{d}}}\beta\right) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} -_{{}^*\mathbb{R}_{\mathbf{d}}} \left(\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta\right)$.
(v) $\forall a \in {}^*\mathbb{R} : (-_{{}^*\mathbb{R}}a)^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \left(-_{{}^*\mathbb{R}_{\mathbf{d}}}\beta\right) = -_{{}^*\mathbb{R}_{\mathbf{d}}} \left(a^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta\right)$.
(vi) $\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} (-_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} 0_{{}^*\mathbb{R}_{\mathbf{d}}}$.

Lemma 1.3.1.4.[24].
(i) $\forall a, b \in {}^*\mathbb{R} : (a \times_{{}^*\mathbb{R}} b)^{\#} = a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#}$.
(ii) Multiplication $(\cdot \times_{{}^*\mathbb{R}} \cdot)$ is associative and commutative:
$\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta = \beta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha, (\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta) \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma = \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma)$.
(iii) $1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = \alpha$; $-1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = -_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha$, where $1_{{}^*\mathbb{R}_{\mathbf{d}}} = (1_{{}^*\mathbb{R}})^{\#}$.
(iv) $|\alpha| \times_{{}^*\mathbb{R}_{\mathbf{d}}} |\beta| = |\beta| \times_{{}^*\mathbb{R}_{\mathbf{d}}} |\alpha|$.

(v) $[(\alpha \geq 0) \wedge (\beta \geq 0) \wedge (\gamma \geq 0)] \implies$
$\implies \alpha \times_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma) = \alpha \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} \alpha \times_{^*\mathbb{R}_\mathbf{d}} \gamma.$
(vi) $0_{^*\mathbb{R}_\mathbf{d}} <_{^*\mathbb{R}_\mathbf{d}} \alpha <_{^*\mathbb{R}_\mathbf{d}} \alpha', 0_{^*\mathbb{R}_\mathbf{d}} <_{^*\mathbb{R}_\mathbf{d}} \beta <_{^*\mathbb{R}_\mathbf{d}} \beta' \implies$
$\implies \alpha \times_{^*\mathbb{R}_\mathbf{d}} \beta <_{^*\mathbb{R}_\mathbf{d}} \alpha' \times_{^*\mathbb{R}_\mathbf{d}} \beta'.$
Proof.(v) Clearly $\alpha \times_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma) \leq \alpha \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} \alpha \times_{^*\mathbb{R}_\mathbf{d}} \gamma$.
Suppose $d \in \alpha \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} \alpha \times_{^*\mathbb{R}_\mathbf{d}} \gamma$.Hence:
$d = ab + a'c$,where $a, a' \in \alpha, b \in \beta, c \in \gamma$.
Without loss of generality we may assume $a \leq a'$.Hence:

$d = ab + a'c \leq a'b + a'c = a'(b+c) \in \alpha \times_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma)$.
Definition 1.3.1.6. Suppose $\alpha \in {}^*\mathbb{R}_\mathbf{d}, 0 <_{^*\mathbb{R}_\mathbf{d}} \alpha$ then $\alpha^{-1_{^*\mathbb{R}_\mathbf{d}}}$ is defined as follows:
(i) $0_{^*\mathbb{R}_\mathbf{d}} <_{^*\mathbb{R}_\mathbf{d}} \alpha : \alpha^{-1_{^*\mathbb{R}_\mathbf{d}}} \triangleq \inf \left\{ a^{-1_{^*\mathbb{R}}} | a \in \alpha \right\}$,
(ii) ${}_{^*\mathbb{R}_\mathbf{d}}\alpha <_{^*\mathbb{R}_\mathbf{d}} 0 : \alpha^{-1_{^*\mathbb{R}_\mathbf{d}}} \triangleq -_{^*\mathbb{R}_\mathbf{d}} (-_{^*\mathbb{R}_\mathbf{d}}\alpha)^{-1_{^*\mathbb{R}_\mathbf{d}}}$.

Lemma 1.3.1.5.[24].
(i) $\forall a \in {}^*\mathbb{R} : \left(a^{\#}\right)^{-1_{^*\mathbb{R}_\mathbf{d}}} = \left(a^{-1_{^*\mathbb{R}}}\right)^{\#}$.
(ii) $\left(\alpha^{-1_{^*\mathbb{R}}}\right)^{-1_{^*\mathbb{R}}} = \alpha.$
(iii) $0_{^*\mathbb{R}_\mathbf{d}} <_{^*\mathbb{R}_\mathbf{d}} \alpha \leq_{^*\mathbb{R}_\mathbf{d}} \beta \implies \beta^{-1_{^*\mathbb{R}_\mathbf{d}}} \leq_{^*\mathbb{R}_\mathbf{d}} \alpha^{-1_{^*\mathbb{R}_\mathbf{d}}}$.
(iv) $[(0_{^*\mathbb{R}_\mathbf{d}} <_{^*\mathbb{R}_\mathbf{d}} \alpha) \wedge (0_{^*\mathbb{R}_\mathbf{d}} <_{^*\mathbb{R}_\mathbf{d}} \beta)] \implies$
$\implies \left(\alpha^{-1_{^*\mathbb{R}}}\right) \times_{^*\mathbb{R}_\mathbf{d}} \left(\beta^{-1_{^*\mathbb{R}}}\right) \leq_{^*\mathbb{R}_\mathbf{d}} (\alpha \times_{^*\mathbb{R}_\mathbf{d}} \beta)^{-1_{^*\mathbb{R}}}$
(v) $\forall a \in {}^*\mathbb{R} : a \neq_{^*\mathbb{R}} 0_{^*\mathbb{R}} \implies \left(\alpha^{\#}\right)^{-1_{^*\mathbb{R}_\mathbf{d}}} \times_{^*\mathbb{R}_\mathbf{d}} \left(\beta^{-1_{^*\mathbb{R}_\mathbf{d}}}\right) = \left(\alpha^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta\right)^{-1_{^*\mathbb{R}_\mathbf{d}}}$.
(vi) $\alpha \times_{^*\mathbb{R}_\mathbf{d}} \alpha^{-1_{^*\mathbb{R}_\mathbf{d}}} \leq_{^*\mathbb{R}_\mathbf{d}} 1_{^*\mathbb{R}_\mathbf{d}}$.

Lemma 1.3.1.5*.Suppose that $a \in {}^*\mathbb{R}, a > 0, \beta, \gamma \in {}^*\mathbb{R}_\mathbf{d}$. Then
$a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma) = a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \gamma.$
Proof. Clearly $a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma) \leq a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \gamma$.
$\left(a^{\#}\right)^{-1_{^*\mathbb{R}}} \left(a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \gamma\right) \leq$
$\leq \left(a^{\#}\right)^{-1_{^*\mathbb{R}}} \left(a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta\right) +_{^*\mathbb{R}_\mathbf{d}} \left(a^{\#}\right)^{-1_{^*\mathbb{R}}} \left(a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \gamma\right) =$
$= \beta +_{^*\mathbb{R}_\mathbf{d}} \gamma$.Thus $\left(a^{\#}\right)^{-1_{^*\mathbb{R}}} \left(a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \gamma\right) \leq \beta +_{^*\mathbb{R}_\mathbf{d}} \gamma$ and one obtain $a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \beta +_{^*\mathbb{R}_\mathbf{d}} a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} \gamma \leq a^{\#} \times_{^*\mathbb{R}_\mathbf{d}} (\beta +_{^*\mathbb{R}_\mathbf{d}} \gamma)$.
Lemma 1.3.1.4. (**General Strong Approximation Property**).
If A is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded from above, then $\sup(A)$ is the unique number such that:
(**i**) $\sup(A)$ is an upper bound for A and
(**ii**) for any $\alpha \in \sup(A)$ there exists $x \in A$ such that $\alpha < x \leq \sup(A)$.
Proof. If not, then α is an upper bound of A less than the least upper bound $\sup(A)$, which is a contradiction.
Lemma 1.3.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be nonempty subsets of ${}^*\mathbb{R} \subset {}^*\mathbb{R}_\mathbf{d}$ and $\mathbf{C} = \{a + b : a \in \mathbf{A}, b \in \mathbf{B}\}$.If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence $\sup(\mathbf{A})$ and $\sup(\mathbf{B})$ exist, then **s**-$\sup(\mathbf{C})$ exist and

$\sup(\mathbf{C}) = \mathrm{su}$

Proof. Suppose $c < \sup(\mathbf{A}) + \sup(\mathbf{B})$.From **Lemma 1.3.1.2.(iv)** ${}^*\mathbb{R}$ is

dense in ${}^*\mathbb{R}_{\mathbf{d}}$.So there is exists $x \in {}^*\mathbb{R}$ such that $c < x^{\#} < \sup(\mathbf{A}) + \sup(\mathbf{B})$.
Suppose that $\alpha, \beta \in {}^*\mathbb{R}$ and $\alpha^{\#} < \sup(\mathbf{A}), \beta^{\#} < \sup(\mathbf{B})$.From **Lemma 1.3.1.4**
(**General Strong Approximation Property**)one obtain there is exists $a \in \mathbf{A}, b \in \mathbf{B}$ such that $\alpha^{\#} < a < \sup(\mathbf{A}), \beta^{\#} < b < \sup(\mathbf{B})$. Suppose $x^{\#} < \alpha^{\#} + \beta^{\#}$.Thus one obtain:

$$\alpha^{\#} - \frac{\left(\alpha^{\#} + \beta^{\#}\right) - x^{\#}}{2} < \alpha^{\#} < a < \sup(\mathbf{A})$$

and

$$\beta^{\#} - \frac{\left(\alpha^{\#} + \beta^{\#}\right) - x^{\#}}{2} < \beta^{\#} < b < \sup(\mathbf{B}).$$

So one obtain

$$x^{\#} = \left[\left(\alpha^{\#} - \frac{\left(\alpha^{\#} + \beta^{\#}\right) - x^{\#}}{2}\right) + \left(\beta^{\#} - \frac{\left(\alpha^{\#} + \beta^{\#}\right) - x^{\#}}{2}\right)\right] < \alpha^{\#} + \beta^{\#} <$$

$$< a + b < \sup(\mathbf{A}) + \sup(\mathbf{B}).$$

But $a + b \in \mathbf{C}$,hence by using **Lemma 1.3.1.4** one obtain that $\sup(\mathbf{C}) = \sup(\mathbf{A}) + \sup(\mathbf{B})$.

Theorem 1.3.1.2. Let $\mathbf{A}$ and $\mathbf{B}$ be nonempty subsets of ${}^*\mathbb{R}_{\mathbf{d}}$ and $\mathbf{C} = \{a + b : a \in \mathbf{A}, b \in \mathbf{B}\}$.If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence $\sup(\mathbf{A})$ and $\sup(\mathbf{B})$ exist, then **s**-$\sup(\mathbf{C})$ exist and $\sup(\mathbf{C}) = \text{su}$

Proof. Suppose $c < \sup(\mathbf{A}) + \sup(\mathbf{B})$.From **Lemma 1.3.1.2.(iv)** ${}^*\mathbb{R}$ is
dense in ${}^*\mathbb{R}_{\mathbf{d}}$.So there is exists $x \in {}^*\mathbb{R}$ such that $c < x^{\#} < \sup(\mathbf{A}) + \sup(\mathbf{B})$.
Suppose that $\alpha, \beta \in {}^*\mathbb{R}$ and $\alpha^{\#} < \sup(\mathbf{A}), \beta^{\#} < \sup(\mathbf{B})$.From **Lemma 1.3.1.4**
(**General Strong Approximation Property**)one obtain there is exists $a \in \mathbf{A}, b \in \mathbf{B}$ such that $\alpha^{\#} < a < \sup(\mathbf{A}), \beta^{\#} < b < \sup(\mathbf{B})$. Suppose $x^{\#} < \alpha^{\#} + \beta^{\#}$.Thus one obtain:

$$\alpha^{\#} - \frac{\left(\alpha^{\#} + \beta^{\#}\right) - x^{\#}}{2} < \alpha^{\#} < a < \sup(\mathbf{A})$$

and

$$\beta^{\#} - \frac{\left(\alpha^{\#} + \beta^{\#}\right) - x^{\#}}{2} < \beta^{\#} < b < \sup(\mathbf{B}).$$

So one obtain

$$x^{\#}=\left[\left(\alpha^{\#}-\frac{\left(\alpha^{\#}+\beta^{\#}\right)-x^{\#}}{2}\right)+\left(\beta^{\#}-\frac{\left(\alpha^{\#}+\beta^{\#}\right)-x^{\#}}{2}\right)\right]<\alpha^{\#}+\beta^{\#}<$$

$$<a+b<\sup(\mathbf{A})+\sup(\mathbf{B}).$$

But $a+b\in\mathbf{C}$,hence by using **Lemma 1.3.1.4** one obtain that $\sup(\mathbf{C})=\sup(\mathbf{A})+\sup(\mathbf{B})$.

Theorem 1.3.1.3. Suppose that $\mathbf{S}$ is a non-empty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded or hyperbounded from above,i.e. $\sup(\mathbf{S})$ exist and suppose that $\xi\in{}^*\mathbb{R},\xi>0$.

Then

$$\sup_{x\in\mathbf{S}}\left\{\xi^{\#}\times x\right\}=\xi^{\#}\times\left(\sup_{x\in\mathbf{S}}\{x\}\right)=\xi^{\#}\times(\sup\mathbf{S}). \tag{1.3.3.2}$$

Proof. Let $B=\mathbf{s}\text{-}\sup\mathbf{S}$.Then B is the smallest number such that, for any $x\in\mathbf{S}$,$x\leq\mathbf{B}$.Let $\mathbf{T}=\left\{\xi^{\#}\times x|x\in\mathbf{S}\right\}$.Since $\xi^{\#}>0,\xi^{\#}\times x\leq\xi^{\#}\times B$ for any $x\in\mathbf{S}$.Hence $\mathbf{T}$ is bounded or hyperbounded above by $\xi^{\#}\times B$.Hence $\mathbf{T}$ has a supremum $C_{\mathbf{T}}=\mathbf{s}\text{-}\sup\mathbf{T}$. Now we have to pruve that $C_{\mathbf{T}}=\xi^{\#}\times B=$ $=\xi^{\#}\times(\sup\mathbf{S})$.Since $\xi^{\#}\times B=\xi^{\#}\times(\sup\mathbf{S})$ is an apper bound for $\mathbf{T}$and C is the smollest apper bound for $\mathbf{T}$,$C_{\mathbf{T}}\leq\xi^{\#}\times B$.Now we repeat the argument above with the roles of $\mathbf{S}$ and $\mathbf{T}$ reversed. We know that $C_{\mathbf{T}}$ is the smallest number such that, for any $y\in\mathbf{T}$,$y\leq C_{\mathbf{T}}$.Since $\xi>0$ it follows that $\left(\xi^{\#}\right)^{-1}\times y\leq\left(\xi^{\#}\right)^{-1}\times C_{\mathbf{T}}$ for any $y\in\mathbf{T}$.But $\mathbf{S}=\left\{\left(\xi^{\#}\right)^{-1}\times y|y\in\mathbf{T}\right\}$.Hence $\left(\xi^{\#}\right)^{-1}\times C_{\mathbf{T}}$ is an apper bound for $\mathbf{S}$.But B is a supremum for $\mathbf{S}$.Hence $B\leq\left(\xi^{\#}\right)^{-1}\times C_{\mathbf{T}}$ and $\xi^{\#}\times B\leq C_{\mathbf{T}}$.We have shown that $C_{\mathbf{T}}\leq\xi^{\#}\times B$ and also that $\xi^{\#}\times B\leq C_{\mathbf{T}}$.Thus $\xi^{\#}\times B=C_{\mathbf{T}}$.

Theorem 1.3.1.4. Suppose that $\alpha\in{}^*\mathbb{R}\ \alpha>0,\beta\in{}^*\mathbb{R}_{\mathbf{d}},\gamma\in{}^*\mathbb{R}_{\mathbf{d}}$.Then

$$\alpha^{\#}\times_{{}^*\mathbb{R}_{\mathbf{d}}}\left(\beta+_{{}^*\mathbb{R}_{\mathbf{d}}}\gamma\right)=\alpha^{\#}\times_{{}^*\mathbb{R}_{\mathbf{d}}}\beta+_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha^{\#}\times_{{}^*\mathbb{R}_{\mathbf{d}}}\gamma. \tag{1.3.3.3}$$

Proof. Let us consider any two sets $S_{\beta}\subset{}^*\mathbb{R}$ and $S_{\gamma}\subset{}^*\mathbb{R}$ such that: $\beta=\sup(S_{\beta}),\gamma=\sup(S_{\gamma})$.Thus by using **Theorem 1.3.1.3** and **Theorem 1.3.1.2** one obtain:

$$\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} (\beta +_{{}^*\mathbb{R}_\mathbf{d}} \gamma) = \alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \sup (S_\beta + S_\gamma) =$$

$$= \sup \left[\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} (S_\beta + S_\gamma)\right] = \sup \left[\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} S_\beta + \alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} S_\gamma\right] =$$

$$= \sup \left(\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} S_\beta\right) + \sup \left(\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} S_\gamma\right) =$$

$$\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \sup (S_\beta) + \alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \sup (S_\gamma).$$

Theorem 1.3.1.5. Suppose that $\alpha \in {}^*\mathbb{R}, \alpha < 0, \beta \in {}^*\mathbb{R}, \gamma \in {}^*\mathbb{R}_\mathbf{d}$.Then

$$\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \left(\beta^{\#} +_{{}^*\mathbb{R}_\mathbf{d}} \gamma\right) = (-1_{{}^*\mathbb{R}_\mathbf{d}}) \times_{{}^*\mathbb{R}_\mathbf{d}} \left[\left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} \beta^{\#} +_{{}^*\mathbb{R}_\mathbf{d}} \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} \gamma\right]. \tag{1.3.3.4}$$

Proof. Let us consider any set $S_\gamma \subset {}^*\mathbb{R}$ such that:$\gamma = \sup (S_\gamma)$.Thus by using **Theorem 1.3.1.3, Theorem 1.3.1.2** and **Lemma 1.3.1.3 (v)** one obtain:

$$\alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \left(\beta^{\#} +_{{}^*\mathbb{R}_\mathbf{d}} \gamma\right) = \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} (-1_{{}^*\mathbb{R}_\mathbf{d}}) \times_{{}^*\mathbb{R}_\mathbf{d}} \left(\beta^{\#} +_{{}^*\mathbb{R}_\mathbf{d}} \gamma\right) =$$

$$= \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} \left[\left(-_{{}^*\mathbb{R}_\mathbf{d}} \beta^{\#}\right) +_{{}^*\mathbb{R}_\mathbf{d}} (-_{{}^*\mathbb{R}_\mathbf{d}} \gamma)\right] =$$

$$= \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} \left(-_{{}^*\mathbb{R}_\mathbf{d}} \beta^{\#}\right) +_{{}^*\mathbb{R}_\mathbf{d}} \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} (-_{{}^*\mathbb{R}_\mathbf{d}} \gamma) =$$

$$= \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} (-1_{{}^*\mathbb{R}_\mathbf{d}}) \times_{{}^*\mathbb{R}_\mathbf{d}} \beta^{\#} +_{{}^*\mathbb{R}_\mathbf{d}} \left|\alpha^{\#}\right| \times_{{}^*\mathbb{R}_\mathbf{d}} (-1_{{}^*\mathbb{R}_\mathbf{d}}) \times_{{}^*\mathbb{R}_\mathbf{d}} \gamma =$$

$$= \alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \beta^{\#} +_{{}^*\mathbb{R}_\mathbf{d}} \alpha^{\#} \times_{{}^*\mathbb{R}_\mathbf{d}} \gamma.$$

13 I.3.2.The topology of ${}^*\mathbb{R}_\mathbf{d}$.Wattenberg standard part.

Fortunately topologically, ${}^*\mathbb{R}_\mathbf{d}$ has many properties strongly reminiscent of $\mathbb{R}$ itself. We proceed as follows [24].

Definition 1.3.2.1.

(i) $(\alpha, \beta)_{{}^*\mathbb{R}_\mathbf{d}} \triangleq \{u | \alpha <_{{}^*\mathbb{R}_\mathbf{d}} u <_{{}^*\mathbb{R}_\mathbf{d}} \beta\}$,

(ii) $[\alpha, \beta]_{{}^*\mathbb{R}_\mathbf{d}} \triangleq \{u | \alpha \leq_{{}^*\mathbb{R}_\mathbf{d}} u \leq_{{}^*\mathbb{R}_\mathbf{d}} \beta\}$.

Definition 1.3.2.2.[24].Suppose $U \subseteqq {}^*\mathbb{R}_\mathbf{d}$. Then U is open if and only if for every $u \in U$, $\exists \alpha_{\alpha \in {}^*\mathbb{R}_\mathbf{d}} \exists \beta_{\beta \in {}^*\mathbb{R}_\mathbf{d}} [\alpha <_{{}^*\mathbb{R}_\mathbf{d}} u <_{{}^*\mathbb{R}_\mathbf{d}} \beta]$ such that $u \in (\alpha, \beta)_{{}^*\mathbb{R}_\mathbf{d}} \subseteqq U$.

Remark.1.3.2.1.[24]. Notice this is not equivalent to: $\forall u_{u \in U} \exists \varepsilon_{\varepsilon > 0} \left[(u - \varepsilon, u + \varepsilon)_{{}^*\mathbb{R}_\mathbf{d}} \subseteqq U\right]$.

Lemma 1.3.2.1.[24].
(i) ${}^*\mathbb{R}$ is dense in ${}^*\mathbb{R}_{\mathbf{d}}$.
(ii) ${}^*\mathbb{R}_{\mathbf{d}} \backslash {}^*\mathbb{R}$ is dense in ${}^*\mathbb{R}_{\mathbf{d}}$.

Lemma 1.3.2.2.[24]. Suppose $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$.Then A is closed if and only if:
(i) $\forall E\,(E \subsetneqq A)$ E bounded above implies $\sup(E) \in A$, and

(ii) $\forall E\,(E \subsetneqq A)$ E bounded below implies $\inf(E) \in A$.
Proposition 1.3.2.1.[24].
(i) ${}^*\mathbb{R}_{\mathbf{d}}$ is connected.
(ii) For $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ in ${}^*\mathbb{R}_{\mathbf{d}}$ set $[\alpha, \beta]_{{}^*\mathbb{R}_{\mathbf{d}}}$ is compact.
(iii) Suppose $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$.Then A is compact if and only if A is closed and bounded.
(iv) ${}^*\mathbb{R}_{\mathbf{d}}$ is normal.
(v) The map $\alpha \longmapsto -_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha$ is continuous.
(vi) The map $\alpha \longmapsto \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$ is continuous.
(vii) The maps $(\alpha, \beta) \longmapsto (\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)$ and $(\alpha, \beta) \longmapsto (\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)$ are not continuous.

Definition 1.3.2.3.[24].(**Wattenberg Standard Part**)
(i) Suppose $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})_{{}^*\mathbb{R}_{\mathbf{d}}}$.Then there is a unique standard $x \in \mathbb{R}$ called $WST(\alpha)$, such that $x \in [\alpha - \varepsilon_{\mathbf{d}}, \alpha + \varepsilon_{\mathbf{d}}]_{{}^*\mathbb{R}_{\mathbf{d}}}$,
(ii) $\alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ implies $WST(\alpha) \leq WST(\beta)$,
(iii) the map $WST(\cdot) : {}^*\mathbb{R}_{\mathbf{d}} \to \mathbb{R}$ is continuous,
(iv) $WST(\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta) = WST(\alpha) + WST(\beta)$,
(v) $WST(\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta) = WST(\alpha) \times WST(\beta)$,
(vi) $WST(-_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha) = -WST(\alpha)$,
(vii) $WST\left(\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) = [WST(\alpha)]^{-1}$ if $\alpha \notin [-\varepsilon_{\mathbf{d}}, \varepsilon_{\mathbf{d}}]_{{}^*\mathbb{R}_{\mathbf{d}}}$.

Proposition 1.3.2.2.[24].Suppose $f : [a, b] \to A \subsetneqq {}^*\mathbb{R}$ is internal, $*$-continuous, and monotonic. Then
(**1**) f has a unique continuous extension $f^{\#}$ $[a, b]_{{}^*\mathbb{R}_{\mathbf{d}}}$ $\overline{A} \to {}^*\mathbb{R}_{\mathbf{d}}$, where $\overline{A}$ denotes the closure of A in ${}^*\mathbb{R}_{\mathbf{d}}$.
(**2**)The conclusion (**1**) above holds iff is piecewise monotonic (i.e., the domain can be decomposed into a finite (not $*$-finite) number of intervals on each of which f is monotonic).
Proposition 1.3.2.3.[24].Suppose f, g are $*$-continuous, piecewise monotonic functions then

(i) $f \circ g$ is also and
(ii) $(f \circ g)^{\#} = \left(f^{\#}\right) \circ \left(g^{\#}\right)$.

14 I.3.3.Absorption numbers in ${}^*\mathbb{R}_{\mathbf{d}}$ and idempotents.

15 I.3.3.1.Absorption function and numbers in ${}^*\mathbb{R}_{\mathbf{d}}$.

One of standard ways of defining the completion of ${}^*\mathbb{R}$ involves restricting oneself to subsets a which have the following property $\forall \varepsilon_{\varepsilon>0} \exists x_{x\in\alpha}\ \exists y_{y\in\alpha}\,[y\ -\ x<\varepsilon]$. It is well known that in this case we obtain a field. In fact the proof is essentially the same as the one used in the case of ordinary Dedekind cuts in the development of the standard real numbers, $\varepsilon_{\mathbf{d}}$,of course, does not have the above property because no infinitesimal works.This suggests the introduction of the concept of absorption part $\mathbf{ab.p.}(\alpha)$ of a number α for an element α of ${}^*\mathbb{R}_{\mathbf{d}}$ which, roughly speaking, measures how much a departs from having the above property [23]. We also introduce similar concept of an absorption number $\alpha\,(\mathbf{ab.n.})\,\beta \triangleq \mathbf{ab.n.}(\alpha,\beta)$ (cut) for given element β of ${}^*\mathbb{R}_{\mathbf{d}}$.

Definition 1.3.3.1.1.[23].$\mathbf{ab.p.}(\alpha) \triangleq \{d \geq 0 | \forall x_{x\in\alpha}\,[x+d\in\alpha]\}$.
Example 1.3.3.1.(**i**) $\forall\alpha\in{}^*\mathbb{R} : \mathbf{ab.p.}(\alpha)=0$,
(**ii**) $\mathbf{ab.p.}(\varepsilon_{\mathbf{d}})=\varepsilon_{\mathbf{d}}$, (**iii**) $\mathbf{ab.p.}(-\varepsilon_{\mathbf{d}})=\varepsilon_{\mathbf{d}}$,
(**iv**) $\forall\alpha\in{}^*\mathbb{R} : \mathbf{ab.p.}(\alpha+\varepsilon_{\mathbf{d}})=\varepsilon_{\mathbf{d}}$,
(**v**) $\forall\alpha\in{}^*\mathbb{R} : \mathbf{ab.p.}(\alpha-\varepsilon_{\mathbf{d}})=\varepsilon_{\mathbf{d}}$.
Definition 1.3.3.2. $\mathbf{ab.n.}(\alpha,\beta) \iff \alpha+\beta=\alpha$.
Example 1.3.3.2.(**i**) $\forall\beta\approx 0 : \mathbf{ab.n.}(\varepsilon_{\mathbf{d}},\beta)$,
(**ii**) $\mathbf{ab.n.}(\varepsilon_{\mathbf{d}},\varepsilon_{\mathbf{d}}), \mathbf{ab.n.}(-\varepsilon_{\mathbf{d}},\varepsilon_{\mathbf{d}}), \mathbf{ab.n.}(-\varepsilon_{\mathbf{d}},-\varepsilon_{\mathbf{d}})$,
(**iii**) $\forall\alpha\in\mathbb{R} : \mathbf{ab.n.}(\alpha+\varepsilon_{\mathbf{d}},\varepsilon_{\mathbf{d}}), \mathbf{ab.n.}(\alpha-\varepsilon_{\mathbf{d}},\varepsilon_{\mathbf{d}}), \mathbf{ab.n.}(\alpha-\varepsilon_{\mathbf{d}},-\varepsilon_{\mathbf{d}})$,
(**iv**) $\forall\alpha\in\mathbb{R} : \mathbf{ab.n.}(\Delta_{\mathbf{d}},\beta)$,

(**v**) $\mathbf{ab.n.}(\Delta_{\mathbf{d}},\Delta_{\mathbf{d}}), \mathbf{ab.n.}(-\Delta_{\mathbf{d}},\Delta_{\mathbf{d}}), \mathbf{ab.n.}(-\Delta_{\mathbf{d}},-\Delta_{\mathbf{d}})$.
Lemma 1.3.3.1.[23].(**i**) $c<\mathbf{ab.p.}(\alpha)$ and $0\leq d<c \implies d\in\mathbf{ab.p.}(\alpha)$
(**ii**) $c\in\mathbf{ab.p.}(\alpha)$ and $d\in\mathbf{ab.p.}(\alpha) \implies c+d\in\mathbf{ab.p.}(\alpha)$.
Remark 1.3.3.1. By **Lemma 1.3.2.1** $\mathbf{ab.p.}(\alpha)$ may be regarded as an

element of ${}^*\mathbb{R}_{\mathbf{d}}$ by adding on all negative elements of ${}^*\mathbb{R}_{\mathbf{d}}$ to $\mathbf{ab.p.}(\alpha)$.

Of course if the condition $d\geq 0$ in the definition of $\mathbf{ab.p.}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathbf{ab.p.}(\alpha)$ since $x<y\in\alpha \implies x\in\alpha$.The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathbf{ab.p.}(\alpha)=\{0\}$. We then identify $\mathbf{ab.p.}(\alpha)$ with 0. [$\mathbf{ab.p.}(\alpha)$ becomes $\{x|x<0\}$ which by our

early convention is not in ${}^{*}\mathbb{R}_{\mathbf{d}}$].

Remark 1.3.3.1.2. By **Lemma 1.3.2.1**(**ii**), **ab**.**p**. (α) is idempotent.
Lemma 1.3.3.1.2.[23].
(**i**) **ab**.**p**.(α) is the maximum element $\beta \in {}^{*}\mathbb{R}_{\mathbf{d}}$ such that $\alpha + \beta = \alpha$.
(**ii**) **ab**.**p**.$(\alpha) \leq \alpha$ for $\alpha > 0$.
(**iii**) If α is positive and idempotent then **ab**.**p**.$(\alpha) = \alpha$.
Lemma 1.3.3.1.3.[23]. Let $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ satsify $\alpha > 0$. Then the following are equivalent. In what follows assume $a, b > 0$.
(**i**) α is idempotent,
(**ii**) $a, b \in \alpha \implies a + b \in \alpha$,
(**iii**) $a \in \alpha \implies 2a \in \alpha$,
(**iv**) $\forall n_{n \in \mathbb{N}} \left[a \in \alpha \implies n \cdot a \in \alpha \right]$,
(**v**) $a \in \alpha \implies r \cdot a \in \alpha$, for all finite $r \in {}^{*}\mathbb{R}$.

16 Connection with the value group.

Definition 1.3.3.1.2. We define an equivalence relation on the positive elements of ${}^{*}\mathbb{R}$ as follows: $a \,\tilde{}\, b \iff \dfrac{a}{b}$ and $\dfrac{b}{a}$ are finite.Then the equivalence classes from a linear ordered set. We denote the order relation by $\ll$.

The classes may be regarded as orders of infinity.
The subring of ${}^{*}\mathbb{R}$ consisting of the finite elements is a valuation ring, and the
equivalence classes may also be regarded as elements of the value group.
Condition (**v**) in Lemma 1.3.3.1.3 essentially says that $a \in \alpha$ and $b\tilde{}a \implies$ b$\in \alpha$,
i.e. a may be regarded as a Dedekind cut in the value group.

17 Properties of the Absorption Function.

Theorem 1.3.3.1.1.[23]. $(-\alpha) + \alpha = -\left[\mathbf{ab.p.}(\alpha)\right]$.

Theorem 1.3.3.1.2.[23].**ab**.**p**.$(\alpha + \beta) \geq$ **ab**.**p**.(α).
Theorem 1.3.3.1.3.[23].
(**i**) $\alpha + \beta \leq \alpha + \gamma \implies -\left[\mathbf{ab.p.}(\alpha)\right] + \beta \leq \gamma$.

(**ii**) $\alpha + \beta = \alpha + \gamma \implies -\left[\mathbf{ab.p.}(\alpha)\right] + \beta = \gamma$.

Theorem 1.3.3.1.4.[23].
(**i**) $\mathbf{ab.p.}(-\alpha) = \mathbf{ab.p.}(\alpha)$,
(**ii**) $\mathbf{ab.p.}(\alpha + \beta) = \max\{\mathbf{ab.p.}(\alpha), \mathbf{ab.p.}(\beta)\}$.

We now classify the elements β such that $\alpha + \beta = \alpha$. For positive β we know by Lemma **1.3.3.1.2.**(**i**) that $\alpha + \beta = \alpha$ iff $\beta \leq \mathbf{ab.p.}(\alpha)$.

Theorem 1.3.3.1.5.[23]. Assume $\beta > 0$. If α absorbs $-\beta$ then a abosrbs β.

Theorem 1.3.3.1.6.[23]. Let $0 < \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. Then the following are equivalent
(**i**) α is an idempotent,
(**ii**) $(-\alpha) + (-\alpha) = -\alpha$,
(**iii**) $(-\alpha) + \alpha = -\alpha$.

18 Special Equivalence Relations on ${}^*\mathbb{R}_{\mathbf{d}}$.

Let Δ be a positive idempotent. We define three equivalence relations
$(\circ\, \mathbf{R} \circ), (\circ\, \mathbf{S} \circ)$ and $(\circ\, \mathbf{T} \circ)$ on ${}^*\mathbb{R}_{\mathbf{d}}$.

Definition 1.3.3.1.3.[23].

(**i**) $\alpha \mathbf{R} \beta \,(\operatorname{mod} \Delta) \iff \alpha + \Delta = \beta + \Delta$,
(**ii**) $\alpha \mathbf{S} \beta \,(\operatorname{mod} \Delta) \iff \alpha + (-\Delta) = \beta + (-\Delta)$,

(**iii**) $\alpha \mathbf{T} \beta \,(\operatorname{mod} \Delta) \iff \exists d\,(d \in \Delta)\,[(\alpha \subset \beta + d) \wedge (\beta \subset \alpha + d)]$.
Remark 1.3.3.1.3. To simplify the notation $\operatorname{mod} \Delta$ is omitted when we are dealing with only one Δ. $\mathbf{R}$ and $\mathbf{S}$ are obviously equivalence relations.
$\mathbf{T}$ is an equivalence relation since Δ is idempotent.

Remark 1.3.3.1.4. It is immediate that $\mathbf{R}, \mathbf{S}$ and $\mathbf{T}$ are congruence relations with respect to addition. Also, if $\tilde{}$ stands for either $\mathbf{R}, \mathbf{S}$ or $\mathbf{T}$ then $\alpha < \beta < \gamma$ and $\alpha \,\tilde{}\, \gamma \implies \alpha \,\tilde{}\, \beta$. To see this it is convenient to have the following lemma.
Lemma 1.3.3.1.4.[23]. Suppose $\alpha < \beta$. Then
(**i**) $\alpha \mathbf{R} \beta \,(\operatorname{mod} \Delta) \iff \beta \leq \alpha + \Delta$,

(**ii**) $\alpha \mathbf{S} \beta \,(\operatorname{mod} \Delta) \iff \beta + (-\Delta) \leq \alpha$.
Lemma 1.3.3.1.5.[23]. Let Δ be a positive idempotent. Then
$-[\alpha + (-\Delta)] + (-\Delta) \leq -\alpha$.
Remark 1.3.3.1.5. This is not immediate since the inequality
$(-\alpha) + (-\beta) \leq -(\alpha + \beta)$ goes the wrong way. In fact, this seems surprising at

first since the first addend may be bigger than one intuitively expects, e.g. if

$\alpha = \Delta = \varepsilon_{\mathbf{d}}$ then $-[\alpha + (-\Delta)] = -[\varepsilon_{\mathbf{d}} + (-\varepsilon_{\mathbf{d}})] = \varepsilon_{\mathbf{d}} > 0$. However, $\varepsilon_{\mathbf{d}} + (-\varepsilon_{\mathbf{d}}) = -\varepsilon_{\mathbf{d}}$, so the inequality is valid after all.

Theorem 1.3.3.1.7.[23].
(**i**) **S** is a congruence relation with respect to negation.
(**ii**) **T** is a congruence relation with respect to negation.

(**iii**) **R** is not a congruence relation with respect to negation.
Theorem 1.3.3.1.8.[23]. $\alpha + \Delta$ is the maximum element β satisfying $\beta \mathbf{R} \boldsymbol{\alpha}$.
Theorem 1.3.3.1.9.[23]. $\alpha + (-\Delta)$ is the minimum element β satisfying $\beta \mathbf{S} \boldsymbol{\alpha}$.

Theorem 1.3.3.1.9.[23]. $\mathbf{T} \subsetneqq \mathbf{R} \subsetneqq \mathbf{S}$. Both inclusions are proper.
Theorem 1.3.3.1.10.[23].
(**i**) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then: $\Delta_2 + (-\Delta_1) = \Delta_2$,
(**ii**) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then: $\alpha \mathbf{S} \beta \,(\operatorname{mod} \Delta_1) \implies \alpha \mathbf{R} \beta \,(\operatorname{mod} \Delta_2)$.
Theorem 1.3.3.1.11.[23].Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then $\alpha \mathbf{S} \beta \,(\operatorname{mod} \Delta_1) \implies \alpha \mathbf{T} \beta \,(\operatorname{mod} \Delta_2)$ but not conversely.
$\blacksquare$
Theorem 1.3.3.1.12.[23].**S** is the smallest congruence relation with respect to addition and negation containing **R**.
Theorem 1.3.3.1.13.[23].Any convex congruence relation ($\circ\ \tilde{}\circ$) containing **T**
properly must contain **S**.

19 I.3.3.2.Special Kinds of Idempotents in ${}^*\mathbb{R}_{\mathbf{d}}$.

Let $a \in {}^*\mathbb{R}$ such that $a > 0$. Then a gives rise to two idempotents in a natural way.
Definition 1.3.3.2.1.[23].
(**i**) $\mathbf{A}_a \triangleq \{x | \exists n_{n \in \mathbb{N}} [x < n \cdot a]\}$.
(**ii**) $\mathbf{B}_a \triangleq \left\{x | \forall r_{r \in \mathbb{R}_+} [x < r \cdot a]\right\}$.
Then it is immediate that $\mathbf{A}_a$ and $\mathbf{B}_a$ are idempotents.The usual "$\epsilon/2$ argument"
shows this for $\mathbf{B}_a$.It is also clear that $\mathbf{A}_a$ is the smallest idempotent containing a
and $\mathbf{B}_a$ is the largest idempotent not containing a.It follows that $\mathbf{B}_a$ and $\mathbf{A}_a$ are
consecutive idempotents.

Remark 1.3.3.2.1. Note that $\mathbf{B}_1 = \varepsilon_{\mathbf{d}} = \inf(\mathbb{R}_+)$ (which is the set of all infinite
small positive numbers plus all negative numbers) which we have already

considered above. $\mathbf{A}_1 = \Delta_{\mathbf{d}} \triangleq \sup(\mathbb{R}_+)$ (which is the set of all finite numbers

plus all negative numbers) which we have also already considered above.

Definition 1.3.3.2.2. Let $a \in {}^*\mathbb{N}$.
(**i**) $\boldsymbol{\omega}_{\mathbf{d}}[a] \triangleq \{x | \exists n_{n\in\mathbb{N}} [x < n \cdot a]\}$.
(**ii**) $\boldsymbol{\Omega}_{\mathbf{d}}[a] = \left\{x | \forall r_{r\in\mathbb{R}_+} [x < r \cdot a]\right\}, a \in {}^*\mathbb{N}_\infty$

Remark 1.3.3.2.2. Then it is immediate that $\boldsymbol{\omega}_{\mathbf{d}}[a]$ and $\boldsymbol{\Omega}_{\mathbf{d}}[a]$ are idempotents.
It is also clear that $\boldsymbol{\omega}_{\mathbf{d}}[a]$ is the smallest idempotent containing hypernatural a
and $\boldsymbol{\omega}_{\mathbf{d}}[a] = a \cdot \boldsymbol{\omega}_{\mathbf{d}}$. $\boldsymbol{\Omega}_{\mathbf{d}}[a] = a \cdot \varepsilon_{\mathbf{d}}$ is the largest idempotent not containing a.
It follows that $\boldsymbol{\Omega}_{\mathbf{d}}[a]$ and $\boldsymbol{\omega}_{\mathbf{d}}[a]$ are consecutive idempotents.
Remark 1.3.3.2.3. Note that $\boldsymbol{\omega}_{\mathbf{d}}[1] = \boldsymbol{\omega}_{\mathbf{d}}$ (which is the set of all finite natural
numbers $\mathbb{N}$ plus all negative numbers) which we have also already considered above.

Theorem 1.3.3.2.1. [23].
(**i**) No idempotent of the form $\mathbf{A}_a$ has an immediate successor.
(**ii**) All consecutive pairs of idempotents have the form $\mathbf{A}_a$ and $\mathbf{B}_a$
for some $a \in {}^*\mathbb{R}$.

20 I.3.3.3. Types of α with a given ab.p.(α).

Among elements of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\mathbf{ab.p.}(\alpha) = \Delta$ we can distinguish
two types.

Definition 1.3.3.3.1. [23]. Assume $\Delta > 0$.
(**i**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1 if $\exists x (x \in \alpha) \forall y [x + y \in \alpha \implies y \in \Delta]$,
(**ii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2 if $\forall x (x \in \alpha) \exists y (y \notin \Delta) [x + y \in \alpha]$, i.e.
$\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2 iff α does not have type 1.
A similar classification exists from above.

Definition 1.3.3.3.2. [23]. Assume $\Delta > 0$.

(**i**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1**A** if $\exists x\,(x \notin \alpha)\,\forall y\,[x - y \notin \alpha \implies y \in \Delta]$,
(**ii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2**A** if $\forall x\,(x \notin \alpha)\,\exists y\,(y \notin \alpha)\,[x - y \notin \alpha]$.

Theorem 1.3.3.3.3.[23].
(**i**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1 iff $-\alpha$ has type 1**A**,

(**ii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ cannot have type 1 and type 1**A** simultaneously.
Theorem 1.3.3.3.4.[23].Suppose $\mathbf{ab.p.}(\alpha) = \Delta > 0$. Then α has type 1 iff α has the form $a+\ \Delta$ for some $a \in {}^*\mathbb{R}$.
Theorem 1.3.3.3.5.[23].$\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1**A** iff α has the form $a+\ (-\Delta)$ for some $a \in {}^*\mathbb{R}$.

Theorem 1.3.3.3.6.[23].
(**i**) If $\mathbf{ab.p.}(\alpha) > \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 1 iff α has type 1.
(**ii**) If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 2 iff either α or β has type 2.
Theorem 1.3.3.3.7.[23]. If $\mathbf{ab.p.}(\alpha)$ has the form $\mathbf{B}_a$ then α has type 1 or type 1**A**.

21 I.3.3.4. ε-Part of α with ab.p.$(\alpha) \neq 0$.

Theorem 1.3.3.4.8. (**i**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}}$ i.e. α has type 1.

Then there is exist *unique* $a \in \mathbb{R}$ such that

$$\alpha = ({}^*a)^{\#} + \varepsilon_{\mathbf{d}}$$

$$a = WST\,(\alpha)$$

(**ii**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha_1 < \Delta_{\mathbf{d}}, -\Delta_{\mathbf{d}} < \alpha_2 < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha_1) = \varepsilon_{\mathbf{d}}, \mathbf{ab.p.}(\alpha_2) = \varepsilon_{\mathbf{d}}$ i.e. α_1 and α_2 has type 1.

Then:

$$\alpha_1 + \alpha_2 = WST\,(\alpha_1) + WST\,(\alpha_2) + \varepsilon_{\mathbf{d}}. \tag{1.3.3.6}$$

(**iii**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}}$ i.e. α has type 1.

Then $\forall b\,(b \in\ {}^*\mathbb{R})$:

$$b^{\#} \times \alpha = b^{\#} \times ({}^*WST\,(\alpha))^{\#} + b^{\#} \times \varepsilon_{\mathbf{d}}.$$

.
(**iv**) Suppose:

1) $-\Delta_{\mathbf{d}} < \alpha_1 < \Delta_{\mathbf{d}}, -\Delta_{\mathbf{d}} < \alpha_2 < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha_1) = \varepsilon_{\mathbf{d}}, \mathbf{ab.p.}(\alpha_2) = \varepsilon_{\mathbf{d}}$ i.e. α_1 and α_2 has type 1.

Then $\forall b \, (b \in {}^*\mathbb{R})$:
$$b^{\#} \times (\alpha_1 + \alpha_2) = b^{\#} \times ({}^*WST\,(\alpha))^{\#} + b^{\#} \times ({}^*WST\,(\alpha_2))^{\#} + b^{\#} \times \varepsilon_{\mathbf{d}}.$$

Theorem 1.3.3.4.9. (**i**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha) = -\varepsilon_{\mathbf{d}}$ i.e. α has type $1A$.

Then there is exist *unique* $a \in \mathbb{R}$ such that
$$\alpha = ({}^*a)^{\#} - \varepsilon_{\mathbf{d}}.$$
$$a = WST\,(\alpha)\,.$$

(**ii**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha_1 < \Delta_{\mathbf{d}}, -\Delta_{\mathbf{d}} < \alpha_2 < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha_1) = -\varepsilon_{\mathbf{d}}, \mathbf{ab.p.}(\alpha_2) = -\varepsilon_{\mathbf{d}}$ i.e. α_1 and α_2 has type $1A$ or
3) $\mathbf{ab.p.}(\alpha_1) = \varepsilon_{\mathbf{d}}, \mathbf{ab.p.}(\alpha_2) = -\varepsilon_{\mathbf{d}}$ i.e. α_1 has type 1 and α_2 has type $1A$.Then:
$$\alpha_1 + \alpha_2 = WST\,(\alpha_1) + WST\,(\alpha_2) - \varepsilon_{\mathbf{d}}. \quad (1.3.$$

(**iii**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha) = -\varepsilon_{\mathbf{d}}$ i.e. α has type $1A$.

Then $\forall b \, (b \in {}^*\mathbb{R})$:
$$b^{\#} \times \alpha = b^{\#} \times ({}^*WST\,(\alpha))^{\#} - b^{\#} \times \varepsilon_{\mathbf{d}}. \quad ($$
.
(**iv**) Suppose:
1) $-\Delta_{\mathbf{d}} < \alpha_1 < \Delta_{\mathbf{d}}, -\Delta_{\mathbf{d}} < \alpha_2 < \Delta_{\mathbf{d}}$,
2) $\mathbf{ab.p.}(\alpha_1) = -\varepsilon_{\mathbf{d}}, \mathbf{ab.p.}(\alpha_2) = -\varepsilon_{\mathbf{d}}$ i.e. α_1 and α_2 has type $1A$ or
3) $\mathbf{ab.p.}(\alpha_1) = \varepsilon_{\mathbf{d}}, \mathbf{ab.p.}(\alpha_2) = -\varepsilon_{\mathbf{d}}$ i.e. α_1 has type 1 and α_2 has type $1A$.Then $\forall b \, (b \in {}^*\mathbb{R})$:
$$b^{\#} \times (\alpha_1 + \alpha_2) = b^{\#} \times ({}^*WST\,(\alpha))^{\#} + b^{\#} \times ({}^*WST\,(\alpha_2))^{\#} - b^{\#}$$

Definition 1.3.3.4.3.Suppose $\mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}}$ i.e. α has type 1, i.e. $\alpha = ({}^*a)^{\#} + \varepsilon_{\mathbf{d}}, a = WST\,(\alpha)\,, a \in \mathbb{R}$.

Then $\beta \triangleq [\alpha]_{\varepsilon}\,, (\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R})$ is an ε-part of α iff:
$$\forall y \left[\left(({}^*a)^{\#} + y \in \alpha \right) \wedge \left(({}^*a)^{\#} + y \in \beta \right) \right.$$

Theorem 1.3.3.4.10.Suppose $-\Delta_{\mathbf{d}} < \alpha < \Delta_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}}$ i.e. α has type $1, -\Delta_{\mathbf{d}} < b < \Delta_{\mathbf{d}}, b \in {}^*\mathbb{R}, c \in \mathbb{R}$.
Then
(i) $[\alpha]_{\varepsilon} = ({}^*a)^{\#} + \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$.
(ii) $\left[b^{\#} + \alpha\right]_{\varepsilon} = ({}^*\,(\mathbf{st}\,(b)))^{\#} + ({}^*a)^{\#} + \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$.
(iii) $\left[({}^*c)^{\#} + \alpha\right]_{\varepsilon} = ({}^*\,(c))^{\#} + ({}^*a)^{\#} + \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$.

Theorem 1.3.3.4.11.(i) Suppose $-\Delta_{\mathbf{d}} < \alpha_1 < \Delta_{\mathbf{d}}, -\Delta_{\mathbf{d}} < \alpha_2 < \Delta_{\mathbf{d}}$,
$\mathbf{ab.p.}(\alpha_1) = \mathbf{ab.p.}(\alpha_2) = \varepsilon_{\mathbf{d}}, WST(\alpha_1) = a \in \mathbb{R}, WST(\alpha_2) = b \in \mathbb{R}$.
Then $[\alpha_1 + \alpha_2]_\varepsilon = ({}^*a)^\# + ({}^*b)^\# + \varepsilon^\# \times \varepsilon_{\mathbf{d}}$.
(ii) $[\alpha_1 - \alpha_2]_\varepsilon = ({}^*a)^\# - ({}^*b)^\# - \varepsilon^\# \times \varepsilon_{\mathbf{d}}$.
Theorem 1.3.3.4.11.$\forall\varepsilon\,(\varepsilon \approx 0)\left[\alpha = \varepsilon_{\mathbf{d}} \iff [\alpha]_\varepsilon = \varepsilon^\# \times \varepsilon_{\mathbf{d}}\right]$.
Definition 1.3.3.4.4. (i) Suppose $\mathbf{ab.p.}(\alpha) = \Delta \geq \varepsilon_{\mathbf{d}}, \alpha \neq \Delta$ and α has type 1,i.e. α has representation $\alpha = a^\# + \Delta$ for some $a \in {}^*\mathbb{R}, a^\# \notin \Delta$.
Then $\beta \triangleq \left[\alpha|a^\#\right]_\varepsilon$,($\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}$) is an ε-part of α *for a given* $a \in {}^*\mathbb{R}$ iff:

$$\forall y\left[\left(a^\# + y \in \alpha\right) \wedge \left(a^\# + y \in \beta\right) \iff y \in \varepsilon^\# \times \Delta\right]. \quad (1.3.3.14)$$

(ii) Suppose $\mathbf{ab.p.}(\alpha) = \Delta \geq \varepsilon_{\mathbf{d}}, \alpha = \Delta$.Then $\beta \triangleq [\alpha|\Delta]_\varepsilon$,

$\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}$.

is an ε-part of α *for a given* $a \in {}^*\mathbb{R}$ iff:

$$\forall y\left[(y \in \alpha) \wedge (y \in \beta) \iff y \in \varepsilon^\# \times \Delta\right].$$

Note if $\mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}}$ and $\alpha = ({}^*a)^\# + \varepsilon_{\mathbf{d}}, a \in \mathbb{R}$
then $\beta \triangleq \left[\alpha|\left({}^*a\right)^\#\right]_\varepsilon = [\alpha]_\varepsilon$, $[\alpha]_\varepsilon$is an ε-part of α.

Definition 1.3.3.4.4. Suppose $\mathbf{ab.p.}(\alpha) = \Delta \leq -\varepsilon_{\mathbf{d}}$ and α has type $1A$,i.e.i.e. α has representation $\alpha = a^\# - \Delta$ for some $a \in {}^*\mathbb{R}, a^\# \notin \Delta$.Then $\beta \triangleq \left[\alpha|a^\#\right]_\varepsilon$ is an ε-part of α *for a given*

$a \in {}^*\mathbb{R}$ iff:

$$\forall y\left(a^\# + y \in \alpha\right) \wedge \left(a^\# + y \in \beta\right) \iff y \in \varepsilon^\# \times \Delta. \quad (1.3.3.1$$

Note if $\mathbf{ab.p.}(\alpha) = -\varepsilon_{\mathbf{d}}$ i.e. $\alpha = ({}^*a)^\# - \varepsilon_{\mathbf{d}}, a \in \mathbb{R}$
then $\beta \triangleq \left[\alpha|\left({}^*a\right)^\#\right]_\varepsilon = [\alpha]_\varepsilon$, $[\alpha]_\varepsilon$ is an ε-part of α.

Theorem 1.3.3.3.10.
(**1**) Suppose $\mathbf{ab.p.}(\alpha) = \Delta \geq \varepsilon_{\mathbf{d}}$ and α has type 1,
i.e. $\alpha = a^\# + \Delta$ for some $a \in {}^*\mathbb{R}$.
Then $\left[\alpha|a^\#\right]_\varepsilon$ has the form

$$\left[\alpha|a^\#\right]_\varepsilon = a^\# + \varepsilon^\# \times \Delta$$

for a given $a \in {}^*\mathbb{R}$.
(**2**) Suppose $\mathbf{ab.p.}(\alpha) = \Delta \leq -\varepsilon_{\mathbf{d}}$ and α has type $1A$,
i.e. $\alpha = a^\# - \Delta$ for some $a \in {}^*\mathbb{R}$.
Then $\left[\alpha|a^\#\right]_\varepsilon$ has the form

$$\left[\alpha|a^\#\right]_\varepsilon = a^\# - \varepsilon^\# \times \Delta \quad (1.3.3.18)$$

for a given $a \in {}^*\mathbb{R}$.

Theorem 1.3.3.4.11.
(**1**) Suppose **ab.p.**$(\alpha) = \varepsilon_{\mathbf{d}}$ i.e. α has type 1 and
α has representation $\alpha = ({}^*a)^{\#} + \varepsilon_{\mathbf{d}}$,for some *unique* $a \in \mathbb{R}$.
Then $[\alpha]_\varepsilon$ has the unique form:

$$[\alpha]_\varepsilon = ({}^*a)^{\#} + \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}. \tag{1.3.3.19}$$

(**2**) Suppose **ab.p.**$(\alpha) = -\varepsilon_{\mathbf{d}}$ i.e. α has type $1A$ and
α has representation $\alpha = ({}^*a)^{\#} - \varepsilon_{\mathbf{d}}$, for some *unique* $a \in \mathbb{R}$.
Then $[\alpha]_\varepsilon$ has the unique form:

$$[\alpha]_\varepsilon = ({}^*a)^{\#} - \varepsilon \times \varepsilon_{\mathbf{d}}. \tag{1.3.3.20}$$

Theorem 1.3.3.4.12. (**1**) Suppose **ab.p.**$(\alpha) = \varepsilon_{\mathbf{d}}, WST(\alpha) \geq 0$
i.e. α has type 1 and α has representation $\alpha = ({}^*a)^{\#} + \varepsilon_{\mathbf{d}}$,
for some *unique* $a \in \mathbb{R}_+$.Then for every $M \in {}^*\mathbb{R}_+$

$$\begin{gathered} M \times [\alpha]_\varepsilon = \left[M \times \alpha \middle| M \times ({}^*a)^{\#}\right]_\varepsilon = \\ = M \times ({}^*a)^{\#} + \left(\varepsilon^{\#} \times M\right) \times \varepsilon_{\mathbf{d}}. \end{gathered} \tag{1.3.3.21}$$

(**2**) Suppose **ab.p.**$(\alpha) = \varepsilon_{\mathbf{d}}, WST(\alpha) \leq 0$
and α has type 1 i.e. $\alpha = ({}^*a)^{\#} + \varepsilon_{\mathbf{d}}$, for some *unique* $a \in \mathbb{R}_-$.
Then for every $M \in {}^*\mathbb{R}_+$

$$\begin{gathered} M \times [\alpha]_\varepsilon = \left[M \times \alpha \middle| M \times ({}^*a)^{\#}\right]_\varepsilon = \\ = M \times ({}^*a)^{\#} + \left(\varepsilon^{\#} \times M\right) \times \varepsilon_{\mathbf{d}}. \end{gathered} \tag{1.3.3.22}$$

Theorem 1.3.3.4.13. **(i)** Suppose **ab.p.**$(\alpha) = \varepsilon_{\mathbf{d}}$ i.e. α has type 1.
Then $\alpha = \varepsilon_{\mathbf{d}} \iff \forall y \forall \varepsilon\, (\varepsilon \approx 0) \left[(y \in \alpha) \wedge (y \in [\alpha]_\varepsilon) \iff y \in \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}\right]$.
(ii) Suppose **ab.p.**$(\alpha) = -\varepsilon_{\mathbf{d}}$ i.e. α has type $1A$.
Then $\alpha = -\varepsilon_{\mathbf{d}} \iff \forall y \forall \varepsilon\, (\varepsilon \approx 0) \left[(y \in \alpha) \wedge (y \in [\alpha]_\varepsilon) \iff y \in -\varepsilon^{\#} \times \varepsilon_{\mathbf{d}}\right]$.

22 I.3.3.5.Multiplicative idempotents.

Definition 1.3.3.5.1.[23]. Let $[S]_{\mathbf{d}} = \{x | \exists y\, (y \in S)\, [x \leq y]\}$.Then $[S]_{\mathbf{d}}$ satisfies the usual axioms for a closure operation.

Let f be a continuous strictly increasing function in each variable from a

subset of $\mathbb{R}^n$ into $\mathbb{R}$. Specifically, we want the domain to be the cartesian product $\prod_{i=1}^{n} A_i$, where $A_i = \{x|x > a_i\}$ for some $a_i \in \mathbb{R}$.By transfer f extends to a function ${}^*f$ from the corresponding subset of ${}^*\mathbb{R}^n$ into ${}^*\mathbb{R}$ which is also strictly increasing in each variable and continuous in the Q topology (i.e. ε and δ range over arbitrary positive elements in ${}^*\mathbb{R}$).

Definition 1.3.3.5.2.[23]. Let $\alpha_i \in {}^*\mathbb{R}_{\mathbf{d}}, b_i \in {}^*\mathbb{R}$, then

$$[f]_{\mathbf{d}}(\alpha_1, \alpha_2, ..., \alpha_n) = [\{{}^*f(b_1, b_2, ..., b_n) \mid b_i \in \alpha_i\}]_{\mathbf{d}}$$

Theorem 1.3.3.5.1.[23]. If f and g are functions of one variable then $[f \cdot g]_{\mathbf{d}}(\alpha) = ([f]_{\mathbf{d}}(\alpha)) \cdot ([g]_{\mathbf{d}}(\alpha))$.

Theorem 1.3.3.5.2.[23].Let f and g be any two terms obtained by compositions of strictly increasing continuous functions possibly containing parameters in ${}^*\mathbb{R}$. Then any relation ${}^*f = {}^*g$ or ${}^*f < {}^*g$ valid in ${}^*\mathbb{R}$ extends to ${}^*\mathbb{R}_{\mathbf{d}}$,i.e. $[f]_{\mathbf{d}}(\alpha) = [g]_{\mathbf{d}}(\alpha)$ or $[f]_{\mathbf{d}}(\alpha) < [g]_{\mathbf{d}}(\alpha)$.

Theorem 1.3.3.5.3.[23].The map $\alpha \longmapsto [\exp]_{\mathbf{d}}(\alpha)$ maps the set of additive idempotents onto the set of all multiplicative idempotents other than 0.

Similarly, multiplicative absorption can be defined and reduced to the study of additive absorption. Incidentally the map $\alpha \longmapsto [\exp]_{\mathbf{d}}(\alpha)$ is essentially the same as the map in [34, Theorem 6] which is the map from the set of ideals onto the set of all prime ideals of the valuation ring consisting of the finite elements of ${}^*\mathbb{R}$.

23 I.3.3.6.Additive monoid of Dedekind hyperreal integers ${}^*\breve{\mathbb{Z}}_{\mathbf{d}}$.

Well-order relation $(\cdot \preceq_{\mathbf{s}} \cdot)$ (or strong well-ordering) on a set S is a total order on S with the property: that every non-empty subset S' of S has a least element in this ordering.The set S together with the well-order relation $\preceq_{\mathbf{s}}$ is then called a (strong) well-ordered set.

The natural numbers of ${}^*\mathbb{N}$ with the well-order relation $(\cdot \leq_{{}^*\mathbb{N}} \cdot)$ are not strong

well-ordered set,for there is no smallest infinite one.

Definition 1.3.3.6.1. Weak well-order relation $(\cdot \preceq_w \cdot)$ (or weak well-ordering) on

a set S is a total order on S with the property:every non-empty subset $S' \subseteqq S$ has

a least element in this ordering or S' has a greatest lower bound $(\inf(S'))$ in this

ordering.The set S together with the weak well-order relation $\preceq_w$is then called a

weak well-ordered set.

The natural numbers of ${}^*\mathbb{N}$ with the well-order relation $(\cdot \leq_{{}^*\mathbb{N}} \cdot)$ are not iven

weak well-ordered set,for there is no $\inf(S')$ in S.

Let us considered completion of the ring ${}^*\mathbb{Z}$.Possible standard completion

of the ring ${}^*\mathbb{Z}$ can be constructed by Dedekind sections. Making a semantic

leap, we now answer the question:"what is a *Dedekind hyperintegers* ?"

Definition 1.3.3.6.2. A **Dedekind hyperinteger** is a cut in ${}^*\mathbb{Z}$.

${}^*\breve{\mathbb{Z}}_{\mathbf{d}} = ({}^*\mathbb{Z}_{\mathbf{d}}, +)$ is the class of all Dedekind **hyperinteger**s $x = A|B, A \subsetneqq {}^*\mathbb{Z}, B \subsetneqq {}^*\mathbb{Z}$

$(x = A, A \subsetneqq^* \mathbb{Z})$.

We will show that in a natural way ${}^*\mathbb{Z}_{\mathbf{d}}$ is a complete ordered additive

monoid containing ${}^*\mathbb{Z}$.

Before spelling out what this means, here are some examples of cuts in ${}^*\mathbb{Z}$.

(**i**)

$A|B = \{n \in {}^*\mathbb{Z} \mid n < 1\} | \{n \in {}^*\mathbb{Z} \mid n \geq 1\}$.

(**ii**)

$A|B = \{n \in {}^*\mathbb{Z} \mid n < \omega\} | \{n \in {}^*\mathbb{Z} \mid n \geq \omega\}$,where $\omega \in {}^*\mathbb{N}_\infty$.

(**iii**)

$A|B = \{n \in {}^*\mathbb{Z} \mid (n \leq 0) \vee (n \in \mathbb{Z}_+)\} | \{n \in {}^*\mathbb{Z} \mid (n \in {}^*\mathbb{N}_\infty)\}$.

(**iv**)

$$A|B = \left\{n \in {}^*\mathbb{Z} \mid (n \leq 0) \vee \left[(n \in {}^*\mathbb{Z}_+) \wedge \left(\bigwedge_{i \in \mathbb{N}} (n \leq \omega + i)\right)\right]\right\} \Bigg| \left\{n \in {}^*\mathbb{Z} \mid (n \in {}^*\mathbb{Z}_+) \wedge \left(\bigwedge_{i \in \mathbb{N}} (n > \omega + i)\right)\right\},$$

where $\omega \in {}^*\mathbb{N}_\infty$.

Remark. 1.3.3.6.1. It is convenient to say that $A|B \in {}^*\mathbb{Z}_{\mathbf{d}}$ is an **integer** (hyperinteger) **cut** in ${}^*\mathbb{Z}$ if it is like the cut in examples (**i**),(**ii**): fore some fixed integer (hyperinteger) number $c \in {}^*\mathbb{Z}$, A is the set of all integer $n \in {}^*\mathbb{Z}$ such that $n < c$ while B is the rest of ${}^*\mathbb{Z}$.

The B-set of an integer (hyperinteger) cut contains a smollest c, and

conversaly if $A|B$ is a cut in ${}^*\mathbb{Z}$ and B contains a smollest element c then $A|B$ is an integer (hyperinteger) cut at c.We write $\breve{c}$ for the integer and hyperinteger cut at c.This lets us think of ${}^*\mathbb{Z} \subset {}^*\mathbb{Z}_{\mathbf{d}}$ by identifying c with $\breve{c}$.

Remark.1.3.3.6.2. It is convenient to say that:

(**1**) $A|B \in {}^*\mathbb{Z}_{\mathbf{d}}$ is an **standard cut** in ${}^*\mathbb{Q}$ if it is like the cut in example (**i**): fore some cut $A'|B' \in \mathbb{Z}$ the next equality is satisfied:$A|B = {}^*(A')\,|^*\,(B')$, i.e. A-set of a cut is an standard set.

(**2**) $A|B \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ is an **internal cut** or **nonstandard cut** in ${}^*\mathbb{Z}$ if it is like the cut in example (**ii**), i.e. A-set of a cut is an *internal nonstandard set.*

(**3**) $A|B \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ is an **external cut** in ${}^*\mathbb{Z}$ if it is like the cut in examples (**iii**)-(**iv**), i.e. A-set of a cut is an *external set.*

Definition 1.3.3.6.3. A Dedekind cut α in ${}^*\mathbb{Z}$ is a subset $\alpha \subset {}^*\mathbb{Z}$ of the hyperinteger numbers ${}^*\mathbb{Z}$ that satisfies these properties:

1. α is not empty.

2. $\beta = {}^*\mathbb{Z}\backslash\alpha$ is not empty.

3. α contains no greatest element.

4. For $x, y \in {}^*\mathbb{Z}$, if $x \in \alpha$ and $y < x$, then $y \in \alpha$ as well.

Definition 1.3.3.6.4. A Dedekind hyperinteger $\alpha \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ is a Dedekind cut α in ${}^*\mathbb{Z}$. We denote the set of all Dedekind hyperinteger by ${}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ and we order them by set-theoretic inclusion, that is to say, for any $\alpha, \beta \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$, $\alpha <_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta$ (or $\alpha < \beta$) if and only if $\alpha \subsetneqq \beta$ where the inclusion is strict. We further define $\alpha =_{{}^*\mathbb{Z}_{\mathbf{d}}} \beta$ (or $\alpha = \beta$) as hyperinteger if and are equal as sets. As usual, we write $\alpha \leqslant_{{}^*\mathbb{Z}_{\mathbf{d}}} \beta$ if $\alpha <_{{}^*\mathbb{Z}_{\mathbf{d}}} \beta$ or $\alpha =_{{}^*\mathbb{Z}_{\mathbf{d}}} \beta$.

Definition 1.3.3.6.5. $M \in {}^*\mathbb{Z}_{\mathbf{d}}$ is an **upper bound** for a set $S \subset {}^*\mathbb{R}_{\mathbf{d}}$ if each $s \in S$ satisfies $s \leq_{{}^*\mathbb{Z}_{\mathbf{d}}} M$. We also say that the set S is **bounded above** by M iff $M \in \mathbf{L}({}^*\mathbb{R})$.We also say that the set S is **hyperbounded above** iff $M \notin \mathbf{L}({}^*\mathbb{R})$,i.e. $|M| \in {}^*\mathbb{R}_+\backslash\mathbb{R}_+$.

Definition 1.3.3.6.6. An upper bound for S that is less than all other upper bound for S is a **least upper bound** for S.

Theorem 1.3.3.6.1 Every nonempty subset $A \subsetneqq {}^*\mathbb{Z}_{\mathbf{d}}$ of Dedekind hyperinteger that is bounded (hyperbounded) above has a least upper bound.

Definition 1.3.3.6.7. Given two Dedekind hyperinteger α and β we define:

1.The additive identity (zero cut) denoted $0_{{}^*\mathbb{Z}_{\mathbf{d}}}$ or **0**,is
$0_{{}^*\mathbb{Z}_{\mathbf{d}}} \triangleq \{x \in {}^*\mathbb{Z}|\ x < 0\}$.

2.The multiplicative identity denoted $1_{{}^*\mathbb{Z}_{\mathbf{d}}}$ or 1,is
$1_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \triangleq \{x \in {}^*\mathbb{Z}|\ x < 1\}$.

3. Addition $\alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta$ of α and β also denoted $\alpha + \beta$ is

$\alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta \triangleq \{x+y|\ x \in \alpha, y \in \beta\}$.

It is easy to see that $\alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} 0_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} = 0_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}}$ for all $\alpha \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$.

It is easy to see that $\alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta$ is a cut in ${}^*\mathbb{Z}$ and $\alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta = \beta +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \alpha$.

Another fundamental property of cut addition is associativity:

$\left(\alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta\right) +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \gamma = \alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \left(\beta +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \gamma\right)$.

This follows from the corresponding property of ${}^*\mathbb{Z}$.

4.The opposite $-_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}}\alpha$ of α, also denoted $-\alpha$,is

$-\alpha \triangleq \{x \in\ {}^*\mathbb{Z}|\ -x \notin \alpha, -x$ is not the least element of ${}^*\mathbb{Z}\backslash\alpha\}$.

5.Remark 1.3.3.6.3. We also say that the opposite $-\alpha$ of α is the **additive inverse** of α denoted $\div\alpha$ iff the next equality is satisfied: $\alpha + (\div\alpha) = 0$.

6.Remark 1.3.3.6.4. It is easy to see that for all standard and internal cut $\alpha^{\mathbf{Int}}$ the

opposite $-\alpha^{\mathbf{Int}}$ is the additive inverse of $\alpha^{\mathbf{Int}}$,i.e. $\alpha^{\mathbf{Int}} + \left(\div\alpha^{\mathbf{Int}}\right) = 0$.

7.We say that the cut α is positive if $0 < \alpha$ or negative if $\alpha < 0$.

The absolute value of α,denoted $|\alpha|$,is $|\alpha| \triangleq \alpha$,if $\alpha \geq 0$ and $|\alpha| \triangleq -\alpha$,if $\alpha \leq 0$.

8. The cut order enjois on ${}^*\mathbb{Z}_{\mathbf{d}}$ the standard additional properties of:

(**i**) **transitivity:** $\alpha \leq_{{}^*\mathbb{Z}_{\mathbf{d}}} \beta \leq_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \gamma \implies \alpha \leq_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \gamma$.

(**ii**) **trichotomy:** eizer $\alpha <_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta, \beta <_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \alpha$ or $\alpha =_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta$ but only one of the three things is true.

(**iii**) **translation:** $\alpha \leq_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta \implies \alpha +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \gamma \leq_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \beta +_{{}^*\breve{\mathbb{Z}}_{\mathbf{d}}} \gamma$.

9.By definition above, this is what we mean when we say that

${}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ is an **complete ordered additive monoid.**

Remark 1.3.3.6.5. Let us considered Dedekind integer cut $c \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ as subset of

$c \subset {}^*\mathbb{R}_{\mathbf{d}}$.We write $\widetilde{c} = {}^*\mathbb{R}_{\mathbf{d}}\text{-}\sup(c) = \sup\limits_{x}\{x|x \in c \subset\ {}^*\mathbb{R}_{\mathbf{d}}\}$ for the cut $c \in {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$.

This lets us think of canonical imbeding ${}^*\breve{\mathbb{Z}}_{\mathbf{d}} \underset{\mathbf{j_d}}{\longmapsto} {}^*\mathbb{R}_{\mathbf{d}}$ monoid

${}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ into generalized pseudo-field ${}^*\mathbb{R}_{\mathbf{d}}$

$${}^*\breve{\mathbb{Z}}_{\mathbf{d}} \subset\longrightarrow\ {}^*\mathbb{R}_{\mathbf{d}}$$

by identifying c with it image $\widetilde{c} = j_{\mathbf{d}}(c)$.

Remark 1.3.3.6.6. It is convenient to identify monoid ${}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ with it image $j_{\mathbf{d}}\left({}^*\breve{\mathbb{Z}}_{\mathbf{d}}\right) \subset {}^*\mathbb{R}_{\mathbf{d}}$.

24 I.3.5.Pseudo-ring of Wattenberg hyperreal integers ${}^*\mathbb{Z}_{\mathbf{d}}$.

The set ${}^*\mathbb{R}_{\mathbf{d}}$ has within it a set ${}^*\mathbb{Z}_{\mathbf{d}} \subsetneqq {}^*\breve{\mathbb{Z}}_{\mathbf{d}}$ of Wattenberg hyperreal integers which behave very much like hyperreals ${}^*\mathbb{Z}$ inside ${}^*\mathbb{R}$.In particular the

greatest integer function ${}^*[\cdot] : {}^*\mathbb{R} \to {}^*\mathbb{Z}$ extends in a natural way to $[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} : {}^*\mathbb{R}_\mathbf{d} \to {}^*\mathbb{Z}_\mathbf{d}$.
Lemma 1.3.5.1.[24].Suppose $\alpha \in {}^*\mathbb{R}_\mathbf{d}$. Then the following two conditions on α are equivalent:
(**i**) $\alpha = \sup\left\{\nu^{\#} | (\nu \in {}^*\mathbb{Z}) \wedge (\nu \leq_{{}^*\mathbb{R}_\mathbf{d}} \alpha)\right\}$,
(**ii**) $\alpha = \inf\left\{\nu^{\#} | (\nu \in {}^*\mathbb{Z}) \wedge (\alpha \leq_{{}^*\mathbb{R}_\mathbf{d}} \nu)\right\}$.
Definition 1.3.5.7. If α satisfies conditions (**i**) or (**ii**) from lemma 1.3.5.1 α is said to be a ${}^*\mathbb{Z}_\mathbf{d}$-integer or Wattenberg (hyperreal) integer.
Lemma 1.3.5.2.[24].(**i**) ${}^*\mathbb{Z}_\mathbf{d}$ is the closure in ${}^*\mathbb{R}_\mathbf{d}$ of ${}^*\mathbb{Z}$,
(**ii**) ${}^*\mathbb{N}_\mathbf{d}$ is the closure in ${}^*\mathbb{R}_\mathbf{d}$ of ${}^*\mathbb{N}$,
(**iii**) both ${}^*\mathbb{Z}_\mathbf{d}$ and ${}^*\mathbb{N}_\mathbf{d}$ are closed with respect to taking sup and inf.
(**iv**) ${}^*\mathbb{N}_\mathbf{d}$ is a weak well-ordered set.
Lemma 1.3.5.3.[24].Suppose that $\lambda, \nu \in {}^*\mathbb{Z}_\mathbf{d}$. Then,
(**i**) $\lambda +_{{}^*\mathbb{R}_\mathbf{d}} \nu \in {}^*\mathbb{Z}_\mathbf{d}$.
(**ii**) $-_{{}^*\mathbb{R}_\mathbf{d}}\lambda \in {}^*\mathbb{Z}_\mathbf{d}$.
(**iii**) $\lambda \times_{{}^*\mathbb{R}_\mathbf{d}} \nu \in {}^*\mathbb{Z}_\mathbf{d}$.

Definition 1.3.5.8. Suppose $\alpha \in {}^*\mathbb{R}_\mathbf{d}$. Then, we define $[\cdot]_{{}^*\mathbb{R}_\mathbf{d}} : {}^*\mathbb{R}_\mathbf{d} \to {}^*\mathbb{Z}_\mathbf{d}$ by: $[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} \triangleq \sup\{\nu | (\nu \in {}^*\mathbb{Z}) \wedge (\nu \leq_{{}^*\mathbb{R}_\mathbf{d}} \alpha)\}$.
Remark 1.3.5.7. There are two possibilities:
(**i**) collection $\{\nu | (\nu \in {}^*\mathbb{Z}) \wedge (\nu \leq_{{}^*\mathbb{R}_\mathbf{d}} \alpha)\}$ has no greatest element. In this case $[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} = \alpha$ since $[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} <_{{}^*\mathbb{R}_\mathbf{d}} \alpha$ implies $\exists a\,(a \in {}^*\mathbb{R})\left[[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} <_{{}^*\mathbb{R}_\mathbf{d}} a <_{{}^*\mathbb{R}_\mathbf{d}} \alpha\right]$. But then $[a]_{{}^*\mathbb{R}_\mathbf{d}} <_{{}^*\mathbb{R}_\mathbf{d}} \alpha$ which implies $[a]_{{}^*\mathbb{R}_\mathbf{d}} +_{{}^*\mathbb{R}_\mathbf{d}} 1_{{}^*\mathbb{R}_\mathbf{d}} <_{{}^*\mathbb{R}_\mathbf{d}} \alpha$ contradicting with $[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} <_{{}^*\mathbb{R}_\mathbf{d}} a \leq_{{}^*\mathbb{R}_\mathbf{d}} [a]_{{}^*\mathbb{R}_\mathbf{d}} +_{{}^*\mathbb{R}_\mathbf{d}} 1_{{}^*\mathbb{R}_\mathbf{d}}$
(**ii**) collection $\{\nu | (\nu \in {}^*\mathbb{Z}) \wedge (\nu \leq_{{}^*\mathbb{R}_\mathbf{d}} \alpha)\}$ has a greatest element,ν.In this case $[\alpha]_{{}^*\mathbb{R}_\mathbf{d}} = \nu \in {}^*\mathbb{N}$.
Definition 1.3.5.9. ${}^*\mathbb{Z}_\mathbf{d}$-integer $\sup(\mathbb{N}) = \inf({}^*\mathbb{N}_\infty)$ we denote $\omega_\mathbf{d}$.
Definition 1.3.5.10. Suppose $\nu \in {}^*\mathbb{N}_\infty$. Then, we define ν-**block** $\mathbf{bk}[\nu]$ as a set of hyper integers such that $\mathbf{bk}[\nu] = \{\nu \pm n | n \in \mathbb{N}\}$.
For $\nu, \lambda \in {}^*\mathbb{N}_\infty$ there are two possibilities:
(**i**) $\nu - \lambda \in \mathbb{Z}$. In this case $\mathbf{bk}[\nu] = \mathbf{bk}[\lambda]$ and we write $\mathbf{bk}[\nu] = \mathbf{bk}[\widetilde{\nu}]$ where $\widetilde{\nu} \in {}^*\mathbb{Z}_+/\mathbb{Z}$.
(**ii**) $|\nu - \lambda| \in {}^*\mathbb{N}_\infty$.In this case $\mathbf{bk}[\nu] \neq \mathbf{bk}[\lambda]$ and $\mathbf{bk}[\widetilde{\nu}] \neq \mathbf{bk}\left[\widetilde{\lambda}\right]$.
Lemma 1.3.5.11. ${}^*\mathbb{N} = \mathbb{N} \cup \left(\bigcup_{\widetilde{\nu}} \mathbf{bk}[\widetilde{\nu}]\right)$.
Proof. Clear by using [25,Chapt.1,section 9].

25 I.3.6.External summation of countable and hyperfinite sequences in ${}^*\mathbb{R}_\mathbf{d}$.

Definition 1.3.6.1. Let $\mathbf{S}_X$ denote the group of permutations of the set X and $\mathbf{H}_X$

denote ultrafilter on the set X. Permutation $\sigma \in \mathbf{S}_X$ is *admissible* iff σ preserv

$\mathbf{H}_X$,i.e. for any $A \in \mathbf{H}_X$ the next condition is satisfied: $\sigma(A) \in \mathbf{H}_X$.

Below we denote by $\widehat{\mathbf{S}}_{X,\mathbf{H}_X}$ the subgroup $\widehat{\mathbf{S}}_{X,\mathbf{H}_X} \subsetneqq \mathbf{S}_X$ of the all admissible

permutations.

Definition 1.3.6.2. Let us consider countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$,such that

(**a**) $\forall n\,(\mathbf{s}_n \geq 0)$ or (**b**) $\forall n\,(\mathbf{s}_n < 0)$ and hyperreal number denoted $[\mathbf{s}_n]$ which

formed from sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ by the law $[\mathbf{s}_n] = \left(\mathbf{s}_0, \mathbf{s}_0 + \mathbf{s}_1, \mathbf{s}_0 + \mathbf{s}_1 + \mathbf{s}_2, ..., \sum_0^i \mathbf{s}_i, ...\right) \in$

Then external sum of the countable sequence $\mathbf{s}_n$ denoted $Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n$

is

$$(a) : Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n \triangleq \inf\left\{\left[\mathbf{s}_{\sigma(n)}\right] \mid \sigma \in \widehat{\mathbf{S}}_{\mathbb{N},\mathbf{H}_{\mathbb{N}}}\right\}, \qquad (1.3.6.3)$$

$$(b) : Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n \triangleq \sup\left\{\left[\mathbf{s}_{\sigma(n)}\right] \mid \sigma \in \widehat{\mathbf{S}}_{\mathbb{N},\mathbf{H}_{\mathbb{N}}}\right\}$$

accordingly.

Example 1.3.6.1. Let us consider countable sequence $\{\mathbf{1}_n\}_{n\in\mathbb{N}}$ such that: $\forall n\,(\mathbf{1}_n = 1)$ .Hence $[\mathbf{1}_n] = (1, 2, 3, ..., i, ...) = \varpi \in {}^*\mathbb{R}$ and using

Eq.(1.3.3) one obtain $$Ext - \sum_{n\in\mathbb{N}} \mathbf{1}_n = \varpi \in {}^*\mathbb{R}. \qquad ($$

Example 1.3.6.2. Let us consider countable sequence $\{\mathbf{1}_n^{\blacktriangledown}\}_{n\in\mathbb{N}}$ such that:

$\{n|\mathbf{1}_n^{\blacktriangledown} = 1\} \in \mathbf{H}_{\mathbb{N}}$.Hence $[\mathbf{1}_n^{\blacktriangledown}] = (1, 2, 3, ..., i, ...)\,(\mathrm{mod}\mathbf{H}_{\mathbb{N}}) = \varpi \in {}^*\mathbb{R}$ and

using Eq.(1.3.3) one obtain

$$Ext - \sum_{n\in\mathbb{N}} \mathbf{1}_n^{\blacktriangledown} = \varpi \in {}^*\mathbb{R}. \qquad (1.3.6.5)$$

Example 1.3.6.3. (**Euler's infinite number** $E^{\#}$). Let us consider countable

sequence $\mathbf{h}_n = n^{-1}$. Hence

$$[\mathbf{h}_n] = \left(1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, ..., 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{i}, ...\right) \in$$

${}^*\mathbb{R}$

and using Eq.(1.3.3) one obtain

$$Ext-\sum_{n=1}^{\infty}\mathbf{h}_n=E^{\#}\in{}^*\mathbb{R}_{\mathbf{d}}.$$

Definition 1.3.6.8. Let us consider countable sequence $\mathbf{s}_n:\mathbb{N}\to\mathbb{R}$ and two subsequences denoted $\mathbf{s}_n^+:\mathbb{N}\to\mathbb{R},\mathbf{s}_n^-:\mathbb{N}\to\mathbb{R}$ which formed from

sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ by the law

$$\mathbf{s}_n^+=\mathbf{s}_n\iff\mathbf{s}_n\geq 0,$$

$$\mathbf{s}_n^+=0\iff\mathbf{s}_n<0$$

and accordingly by the law

$$\mathbf{s}_n^-=\mathbf{s}_n\iff\mathbf{s}_n<0,$$

$$\mathbf{s}_n^-=0\iff\mathbf{s}_n\geq 0$$

Hence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}=\{\mathbf{s}_n^++\mathbf{s}_n^-\}_{n\in\mathbb{N}}$.

Example 1.3.6.4. Let us consider countable sequence

$$\{\mathbf{1}_n^{\pm}\}_{n\in\mathbb{N}}=\{1,-1,1,-1,...,1,-1,...\}. \tag{1.3.6.9}$$

Hence $\{\mathbf{1}_n^{\pm}\}_{n\in\mathbb{N}}=\{\mathbf{1}_n^++\mathbf{1}_n^-\}_{n\in\mathbb{N}}$ where

$$\{\mathbf{1}_n^+\}_{n\in\mathbb{N}}=\{1,0,1,0,...,1,0,...\}$$

$$\{\mathbf{1}_n^-\}_{n\in\mathbb{N}}=\{0,-1,0,-1,...,0,-1,...\}.$$

Definition 1.3.6.9. The external sum of the arbitrary countable

sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ denoted

$$Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n$$

is

$$Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n\triangleq\left(Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n^+\right)+\left(Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n^-\right). \tag{1.3.6.13}$$

Example 1.3.6.5. Let us consider countable sequence (1.3.9) Using Eq.(1.3.3),Eq.(1.3.13) and Eq.(1.3.5) one obtain

$$Ext-\sum_{n\in\mathbb{N}}\mathbf{1}_{n n\in\mathbb{N}}^{\pm}=\left(Ext-\sum_{n\in\mathbb{N}}\mathbf{1}_n^+\right)+\left(Ext-\sum_{n\in\mathbb{N}}\mathbf{1}_n^-\right)= \tag{1.3.14}$$

$$=\varpi-\varpi=0.$$

Definition 1.3.6.10. Let us consider countable sequence $\mathbf{s}_n^{\#}:\mathbb{N}\to{}^*\mathbb{R}_{\mathbf{d}}$,

such that
$(\mathbf{a})\ \forall n\left(\mathbf{s}_n^{\#} \geq 0\right)$ or $(\mathbf{b})\ \forall n\left(\mathbf{s}_n^{\#} < 0\right)$.

Then external sum of the countable sequence $\mathbf{s}_n^{\#}$ denoted $\#Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n^{\#}$

is

$$(a) : \#Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} \mathbf{s}_{\sigma(n)}^{\#} \middle| \ \sigma \in \mathbf{S}_{\mathbb{N}} \right\}, \tag{1.3.6.16}$$

$$(b) : \#Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} \mathbf{s}_{\sigma(n)}^{\#} | \ \middle| \ \sigma \in \mathbf{S}_{\mathbb{N}} \right\}.$$

Definition 1.3.6.11. Let us consider countable sequence $\mathbf{s}_n^{\#} : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{c}}$

and two subsequences denoted ${}^{\#}\mathbf{s}_n^{+} : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{c}}, {}^{\#}\mathbf{s}_n^{-} : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{c}}$ which

formed from sequence $\left\{\mathbf{s}_n^{\#}\right\}_{n\in\mathbb{N}}$ by the law

$$ {}^{\#}\mathbf{s}_n^{+} = \mathbf{s}_n \iff \mathbf{s}_n^{\#} \geq 0,$$
$$ {}^{\#}\mathbf{s}_n^{+} = 0 \iff \mathbf{s}_n^{\#} < 0$$

and accordingly by the law

$$ {}^{\#}\mathbf{s}_n^{-} = \mathbf{s}_n^{\#} \iff \mathbf{s}_n^{\#} < 0,$$
$$ {}^{\#}\mathbf{s}_n^{-} = 0 \iff \mathbf{s}_n^{\#} \geq 0$$

Hence $\left\{\mathbf{s}_n^{\#}\right\}_{n\in\mathbb{N}} = \left\{{}^{\#}\mathbf{s}_n^{+} + {}^{\#}\mathbf{s}_n^{-}\right\}_{n\in\mathbb{N}}$.

Definition 1.3.6.12. The external sum of the arbitrary countable

sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ denoted $\#Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n^{\#}$

is

$$\#Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n^{\#} \triangleq \left(\#Ext - \sum_{n\in\mathbb{N}} \left({}^{\#}\mathbf{s}_n^{+}\right)\right) + \left(\#Ext - \sum_{n\in\mathbb{N}} \left({}^{\#}\mathbf{s}_n^{-}\right)\right). \tag{1.3.6.20}$$

Definition 1.3.6.13. Let us consider an nonempty subset $\mathbf{A} \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ which

is bounded or hyperbounded from above and such that:
$\sup(\mathbf{A}) \pm \varepsilon \neq \sup(\mathbf{A})$ for any $\varepsilon \approx 0$.We call this least upper bound $\sup(\mathbf{A})$

the *strong least upper bound* or *strong supremum*, written as $\mathbf{s}\text{-}\sup(\mathbf{A})$.

Proposition 1.3.6.1. If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded

from above and strong supremum $\mathbf{s}\text{-}\sup(\mathbf{A})$ exist, then:

($\mathbf{1}$) $\mathbf{s}\text{-}\sup(\mathbf{A})$ is the unique number such that $\mathbf{s}\text{-}\sup(\mathbf{A})$ is an upper bound

for $\mathbf{A}$ and $\mathbf{s}\text{-}\sup(\mathbf{A})-\varepsilon$ is not a upper bound for $\mathbf{A}$ for any $\varepsilon \approx 0, \varepsilon > 0$;

($\mathbf{2}$) (**The Strong Approximation Property**) let $\varepsilon \approx 0, \varepsilon > 0$ there exist $x \in \mathbf{A}$

such that $\mathbf{s}\text{-}\sup(\mathbf{A})-\varepsilon < x \leq \mathbf{s}\text{-}\sup(\mathbf{A})$.

Proof.($\mathbf{2}$) If not, then $\mathbf{s}\text{-}\sup(\mathbf{A})-\varepsilon$ is an upper bound of $\mathbf{A}$ less than the least

upper bound, which is a contradiction.

Corollary 1.3.6.1. Let $\mathbf{A}$ be bounded or hyperbounded from above and

non-empty set such that $\mathbf{s}\text{-}\sup(\mathbf{A})$ exist. There is a function

$\alpha(\circ) : {}^*\mathbb{N}_\infty \to {}^*\mathbb{R}_\mathbf{d}$ such that for all $\mathbf{n} \in {}^*\mathbb{N}_\infty$ we have

$\mathbf{s}\text{-}\sup(\mathbf{A}) - \mathbf{n}^{-1} < \alpha(\mathbf{n}) \leq \mathbf{s}\text{-}\sup(\mathbf{A})$

Theorem 1.3.6.2. Let $\mathbf{A}$ be a non-empty set which is bounded or hyperbounded from below. Then the set of lower bounds of $\mathbf{A}$ has a greatest element.

Proof. Let $-\mathbf{A} \triangleq \{-x | x \in \mathbf{A}\}$.We know that ($\mathbf{i}$) $\forall x_{x\in {}^*\mathbb{R}_\mathbf{d}} \forall y_{y\in {}^*\mathbb{R}_\mathbf{d}} (x \leq y \iff -y \leq -x)$.

Let $l_\mathbf{A}$ be a lower bound of $\mathbf{A}$.Then $l_\mathbf{A} \leq x$ for all $x \in \mathbf{A}$.So $-x \leq -l_\mathbf{A}$ for all

$x \in \mathbf{A}$,that is $y \leq l_\mathbf{A}$ for all $y \in -\mathbf{A}$. So $-\mathbf{A}$ is bounded above, and non-empty,

so by the Theorem 1.3.1 $\sup(-\mathbf{A})$ exists.

We shall prove now that: ($\mathbf{ii}$) $-\sup(-\mathbf{A})$ is a lower bound of $\mathbf{A}$,($\mathbf{iii}$) if $l_\mathbf{A}$ is a lower

bound of $\mathbf{A}$ then $l_\mathbf{A} \leq -\sup(-\mathbf{A})$. ($\mathbf{ii}$) if $x \in \mathbf{A}$ then $-x \in -\mathbf{A}$ and so $-x \leq \sup(-\mathbf{A})$

Hence by statement ($\mathbf{i}$) $x \geq -\sup(-\mathbf{A})$ and we see that $-\sup(-\mathbf{A})$ is a lower

bound of $\mathbf{A}$.($\mathbf{iii}$) If $l_\mathbf{A} \leq x$ for all $x \in \mathbf{A}$ then $-l_\mathbf{A} \geq y$ for all $y \in -\mathbf{A}$.Hence $-l_\mathbf{A} \geq \sup(-\mathbf{A})$ by virtue of $\sup(-\mathbf{A})$ being the least upper bound of $-\mathbf{A}$. Finally we obtain: $l_\mathbf{A} \leq -\sup(-\mathbf{A})$.

Definition 1.3.6.14. We call this greatest element a *greatest lower bound*

or *infinum of* $\mathbf{A}$,written is $\inf(\mathbf{A})$.

Definition 1.3.6.15. Let us consider an nonempty subset $\mathbf{A} \subsetneqq {}^*\mathbb{R}_\mathbf{d}$ which

is bounded or hyperbounded from below and such that:

$\inf(\mathbf{A}) \pm \varepsilon \neq \inf(\mathbf{A})$ for any $\varepsilon > 0$, $\varepsilon \approx 0$.

We call this greatest lower bound $\inf(\mathbf{A})$ a *strong greatest lower bound* or *strong infinum*, written is $\mathbf{s}\text{-}\inf(\mathbf{A})$.

Definition 1.3.6.16. Let us consider an nonempty subset $\mathbf{A} \subsetneqq {}^*\mathbb{R}_\mathbf{d}$ which

is bounded or hyperbounded from below and such that:

($\mathbf{1}$) there exist $\varepsilon_0 \approx 0$ such that $\inf(\mathbf{A}) \pm \varepsilon_0 = \inf(\mathbf{A})$,

(**2**) $\inf(\mathbf{A}) \pm \varepsilon \neq \inf(\mathbf{A})$ for any $\varepsilon > 0$ such that $\varepsilon \geq \varepsilon_0 \approx 0$.
We call this greatest lower bound $\inf(\mathbf{A})$ *almost strong greatest lower bound* or *almost strong infinum*, written is $o\mathbf{s}$-$\inf(\mathbf{A})$.

Definition 1.3.6.17. Let us consider an nonempty subset $\mathbf{A} \subsetneqq {}^*\mathbb{R}_\mathbf{d}$ which

is bounded or hyperbounded from below and such that:
(**1**) $\inf(\mathbf{A}) \pm \varepsilon = \inf(\mathbf{A})$ for any $\varepsilon > 0$, $\varepsilon \approx 0$,
(**2**) $\inf(\mathbf{A}) \pm \varepsilon \neq \inf(\mathbf{A})$ for any $\varepsilon > 0$ such that $\varepsilon \not\approx 0$.
We call this greatest lower bound $\inf(\mathbf{A})$ *weak greatest lower bound* or *weak infinum*, written is w-$\inf(\mathbf{A})$.

Definition 1.3.6.18. Let us consider an nonempty subset $\mathbf{A} \subsetneqq {}^*\mathbb{R}_\mathbf{d}$ which

is hyperbounded from below and such that:
(**1**) $\inf(\mathbf{A}) \pm \alpha = \inf(\mathbf{A})$ for any $\alpha > 0$, $\alpha \in \mathbb{R}$,
(**2**) $\inf(\mathbf{A}) \pm \Gamma \neq \inf(\mathbf{A})$ for any $\Gamma > 0$ such that $\Gamma \in {}^*\mathbb{N}_\infty$.
We call this greatest lower bound $\inf(\mathbf{A})$ *ultra weak greatest lower bound* or *ultra weak infinum*, written is uw-$\inf(\mathbf{A})$.

Proposition 1.3.6.2. (**1**) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded
from below and strong infinum $\mathbf{s}$-$\inf(\mathbf{A})$ exist, then:
$\mathbf{s}$-$\inf(\mathbf{A})$ is the unique number such that $\mathbf{s}$-$\inf(\mathbf{A})$ is an upper bound for $\mathbf{A}$ and $\mathbf{s}$-$\inf(\mathbf{A}) + \varepsilon$ is not a lower bound for $\mathbf{A}$ for any $\varepsilon \approx 0, \varepsilon > 0$;
(**2**) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded from above, then:
$\mathbf{s}$-$\sup(\mathbf{A})$ is the unique number such that $\mathbf{s}$-$\sup(\mathbf{A})$ is an upper bound for $\mathbf{A}$
and $\mathbf{s}$-$\sup(\mathbf{A}) - \varepsilon$ is not an upper bound for $\mathbf{A}$ for any $\varepsilon \approx 0, \varepsilon > 0$.

Proposition 1.3.6.3.(a). (**Strong Approximation Property.**)
(**1**) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded (hyperbounded) from above and such that strong supremum $\mathbf{s}$-$\sup(\mathbf{A})$ exist, and let $\varepsilon \approx 0, \varepsilon > 0$ there exist $x \in \mathbf{A}$ such that $\mathbf{s}\text{-}\sup(\mathbf{A}) - \varepsilon < x \leq \mathbf{s}\text{-}\sup(\mathbf{A})$.
(**2**) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded (hyperbounded) from below and such that strong infinum $\mathbf{s}$-$\inf(\mathbf{A})$ exist, and let $\varepsilon \approx 0, \varepsilon > 0$ there exist $x \in \mathbf{A}$ such that $\mathbf{s}\text{-}\inf(\mathbf{A}) \leq x < \mathbf{s}\text{-}\inf(\mathbf{A}) + \varepsilon$.

Proof. (**1**) If not, then $\mathbf{s}$-$\sup(\mathbf{A}) - \varepsilon$ is an upper bound of $\mathbf{A}$ less than the strong upper bound $\mathbf{s}$-$\sup(\mathbf{A})$, which is a contradiction.
(**2**) If not, then $\mathbf{s}$-$\inf(\mathbf{A}) + \varepsilon$ is an lower bound of $\mathbf{A}$ bigger than the strong lower bound $\mathbf{s}$-$\inf(\mathbf{A})$, which is a contradiction.

Proposition 1.3.6.3.(b) (The Almost Strong Approximation Property.)
(**1**) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded (hyperbounded) from above and such that almost strong supremum $o\mathbf{s}$-$\sup(\mathbf{A})$ exist, and let $\varepsilon \approx 0, \varepsilon > 0, o\mathbf{s}\text{-}\sup(\mathbf{A}) \pm \varepsilon \neq o\mathbf{s}\text{-}\sup(\mathbf{A})$ there exist $x \in \mathbf{A}$ such that $o\mathbf{s}\text{-}\sup(\mathbf{A}) - \varepsilon < x \leq o\mathbf{s}\text{-}\sup(\mathbf{A})$.
(**2**) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded (hyperbounded) from below and such that almost strong infinum $o\mathbf{s}$-$\inf(\mathbf{A})$ exist, and let $\varepsilon \approx 0, \varepsilon > 0, o\mathbf{s}\text{-}\inf(\mathbf{A}) \pm \varepsilon \neq o\mathbf{s}\text{-}\inf(\mathbf{A})$ there exist $x \in \mathbf{A}$ such that

$o\mathbf{s}\text{-inf}(\mathbf{A}) \leq x < o\mathbf{s}\text{-inf}(\mathbf{A}) + \varepsilon$.

Proof. ($\mathbf{1}$) If not, then $o\mathbf{s}\text{-sup}(\mathbf{A}) - \varepsilon$ is an upper bound of $\mathbf{A}$ less than the almost strong upper bound $o\mathbf{s}\text{-sup}(\mathbf{A})$, which is a contradiction.

($\mathbf{2}$) If not, then $o\mathbf{s}\text{-inf}(\mathbf{A}) + \varepsilon$ is an lower bound of $\mathbf{A}$ bigger than the almost strong lower bound $o\mathbf{s}\text{-inf}(\mathbf{A})$, which is a contradiction.

Proposition 1.3.6.3.(c) (The Weak Approximation Property.)

($\mathbf{1}$) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded (hyperbounded) from above and such that weak supremum $w\text{-sup}(\mathbf{A})$ exist, and let $\varepsilon \in \mathbb{R}, \varepsilon > 0, w\text{-sup}(\mathbf{A}) \pm \varepsilon \neq w\text{-sup}(\mathbf{A})$ there exist $x \in \mathbf{A}$ such that $w\text{-sup}(\mathbf{A}) - \varepsilon < x \leq w\text{-sup}(\mathbf{A})$.

($\mathbf{2}$) If $\mathbf{A}$ is a nonempty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded (hyperbounded) from below and such that weak infinum $w\text{-inf}(\mathbf{A})$ exist, and let $\varepsilon \in \mathbb{R}, \varepsilon > 0, w\text{-inf}(\mathbf{A}) \pm \varepsilon \neq w\text{-inf}(\mathbf{A})$ there exist $x \in \mathbf{A}$ such that $w\text{-inf}(\mathbf{A}) \leq x < w\text{-inf}(\mathbf{A}) + \varepsilon$.

Proof. ($\mathbf{1}$) If not, then $w\text{-sup}(\mathbf{A}) - \varepsilon$ is an upper bound of $\mathbf{A}$ less than the weak upper bound $w\text{-sup}(\mathbf{A})$, which is a contradiction.

($\mathbf{2}$) If not, then

Corollary 1.3.6.2. ($\mathbf{1}$) Let $\mathbf{A}$ be bounded or hyperbounded from below and non-empty set such that $\mathbf{s}\text{-inf}(\mathbf{A})$ exist. There is a function $\beta(\circ) : {}^*\mathbb{N}_\infty \to {}^*\mathbb{R}_{\mathbf{d}}$ such that for all $\mathbf{n} \in {}^*\mathbb{N}_\infty$ we have $\mathbf{s}\text{-inf}(\mathbf{A}) \leq \beta(\mathbf{n}) < \mathbf{s}\text{-inf}(\mathbf{A}) + \mathbf{n}^{-1}$.

($\mathbf{2}$) Let $\mathbf{A}$ be bounded or hyperbounded from above and non-empty set such that $\mathbf{s}\text{-sup}(\mathbf{A})$ exist. There is a function $\alpha(\circ) : {}^*\mathbb{N}_\infty \to {}^*\mathbb{R}_{\mathbf{d}}$ such that for all $\mathbf{n} \in {}^*\mathbb{N}_\infty$ we have $\mathbf{s}\text{-sup}(\mathbf{A}) - \mathbf{n}^{-1} < \alpha(\mathbf{n}) \leq \mathbf{s}\text{-sup}(\mathbf{A})$.

Example 1.3.6.6. ($\mathbf{a}$) The subset $\left\{\mathbf{n}^{-1}\right\}_{\mathbf{n}\in{}^*\mathbb{N}_\infty} \triangleq \left\{ \frac{1}{\mathbf{n}} \middle| \mathbf{n} \in {}^*\mathbb{N}_\infty \right\} \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ has a *strong greatest lower bound* $\mathbf{s}\text{-inf}\left(\left\{\mathbf{n}^{-1}\right\}_{\mathbf{n}\in{}^*\mathbb{N}_\infty}\right) = 0$ in ${}^*\mathbb{R}_{\mathbf{d}}$.

($\mathbf{b}$) The subset $\left\{n^{-1}\right\}_{\mathbf{n}\in\mathbb{N}} \triangleq \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ has a *greatest lower bound* $\inf\left(\left\{n^{-1}\right\}_{n\in\mathbb{N}}\right)$ in ${}^*\mathbb{R}_{\mathbf{d}}$ but has not a strong greatest lower bound in ${}^*\mathbb{R}_{\mathbf{d}}$.

Example 1.3.6.7. The subset $\mathbb{R} \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ has the *least upper bound* $\sup(\mathbb{R})$ in ${}^*\mathbb{R}_{\mathbf{d}}$ but has not *strong least upper bound* in ${}^*\mathbb{R}_{\mathbf{d}}$.

Example 1.3.6.8. The subset $\mathbf{I}_*$ the all infinitesimal members of the ${}^*\mathbb{R}$, $\mathbf{I}_* \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ has least upper bound $\sup(\mathbf{I}_*)$ in ${}^*\mathbb{R}_{\mathbf{d}}$ but has not *strong least upper bound* in ${}^*\mathbb{R}_{\mathbf{d}}$.

Example 1.3.6.9. The subset $\mathbb{R}_+ \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ has lower bound $\inf(\mathbb{R}_+)$ in ${}^*\mathbb{R}_{\mathbf{d}}$ but has not *strong lower bound* in ${}^*\mathbb{R}_{\mathbf{d}}$.

Example 1.3.6.10. The subset ${}^*\mathbb{R}_+ \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ has lower bound $\inf({}^*\mathbb{R}_+)$ in ${}^*\mathbb{R}_{\mathbf{d}}$ but has not *strong lower bound* in ${}^*\mathbb{R}_{\mathbf{d}}$.

Proposition 1.3.6.4. Let $\mathbf{A}$ and $\mathbf{B}$ be nonempty subsets of ${}^*\mathbb{R}_{\mathbf{d}}$

Theorem 1.3.6.3.A. Let $\mathbf{A}$ and $\mathbf{B}$ be nonempty subsets of ${}^*\mathbb{R}_{\mathbf{d}}$ and $\mathbf{C} = \{a + b : a \in \mathbf{A}, b \in \mathbf{B}\}$.

($\mathbf{1.a}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence and $\mathbf{s}$-sup$(\mathbf{A})$ and $\mathbf{s}$-sup$(\mathbf{B})$ exist, then $\mathbf{s}$-sup$(\mathbf{C})$ exist and

$$\mathbf{s}\text{-}\sup(\mathbf{C}) = \mathbf{s}\text{-}\sup(\mathbf{A}) +$$

($\mathbf{2.a}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from below,hence and $\mathbf{s}$-inf$(\mathbf{A})$ and $\mathbf{s}$-inf$(\mathbf{B})$ exist, then $\mathbf{s}$-inf$(\mathbf{C})$ exist and

$$\mathbf{s}\text{-}\inf(\mathbf{C}) = \mathbf{s}\text{-}\inf(\mathbf{A}) + \mathbf{s}$$

($\mathbf{1.b}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence and $o\mathbf{s}$-sup$(\mathbf{A})$ and $o\mathbf{s}$-sup$(\mathbf{B})$ exist, then $o\mathbf{s}$-sup$(\mathbf{C})$ exist and

$$o\mathbf{s}\text{-}\sup(\mathbf{C}) = o\mathbf{s}\text{-}\sup$$

($\mathbf{2.b}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from below,hence and $o\mathbf{s}$-inf$(\mathbf{A})$ and $o\mathbf{s}$-inf$(\mathbf{B})$ exist, then $o\mathbf{s}$-inf$(\mathbf{C})$ exist and

$$o\mathbf{s}\text{-}\inf(\mathbf{C}) = o\mathbf{s}\text{-}\inf(\mathbf{A}$$

($\mathbf{1.c}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence and w-sup$(\mathbf{A})$ and w-sup$(\mathbf{B})$ exist, then w-sup$(\mathbf{C})$ exist and

$$w\text{-}\sup(\mathbf{C}) = w\text{-}\sup(\mathbf{A}$$

($\mathbf{2.c}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from below,hence and w-inf$(\mathbf{A})$ and w-inf$(\mathbf{B})$ exist,then w-inf$(\mathbf{C})$ exist and

$$w\text{-}\inf(\mathbf{C}) = w\text{-}\inf(\mathbf{A}) +$$

($\mathbf{1.d}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence and uw-sup$(\mathbf{A})$ and uw-sup$(\mathbf{B})$ exist, then uw-sup$(\mathbf{C})$ exist and

$$uw\text{-}\sup(\mathbf{C}) = uw\text{-}$$

($\mathbf{2.d}$) If $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from below,hence and uw-inf$(\mathbf{A})$ and uw-inf$(\mathbf{B})$ exist, then uw-inf$(\mathbf{C})$ exist and

$$uw\text{-}\inf(\mathbf{C}) = uw\text{-}\mathrm{in}$$

Proof. ($\mathbf{1.a}$) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are bounded or hyperbounded from above,hence $\mathbf{s}$-sup$(\mathbf{A})$ and $\mathbf{s}$-sup$(\mathbf{B})$ exist. Let $c \in \mathbf{C}$. Then $c = a + b$ for numbers $a \in \mathbf{A}$ and $b \in \mathbf{B}$. Since $a \leq \mathbf{s}$-sup$(\mathbf{A})$ and $b \leq \mathbf{s}$-sup$(\mathbf{B})$, $c = a + b \leq \mathbf{s}$-sup$(\mathbf{A})$ + $\mathbf{s}$-sup$(\mathbf{B})$. This shows that $\mathbf{s}$-sup$(\mathbf{A}) + \mathbf{s}$-sup$(\mathbf{B})$ is an upper bound for $\mathbf{C}$, in particular, $\mathbf{C}$ is bounded or hyperbounded from above. Given $\varepsilon > 0$, $\mathbf{s}$-sup$(\mathbf{A}) - \varepsilon/2$ is not an a strong upper bound for $\mathbf{A}$ hence there exists $a' \in \mathbf{A}$ such that $\mathbf{s}$-sup$(\mathbf{A}) - \varepsilon/2 < a'$. Similarly, $\mathbf{s}$-sup$(\mathbf{B})$ $-\varepsilon/2$ is not an upper bound for $\mathbf{B}$ and there exists $b' \in \mathbf{B}$ such that $\mathbf{s}$-sup$(\mathbf{B}) - \varepsilon/2 < b'$. So for $c' = a' + b' \in \mathbf{C}$ we have $\mathbf{s}$-sup$(\mathbf{A})$ + $\mathbf{s}$-sup$(\mathbf{B}) - \varepsilon < c'$. This shows that $\mathbf{s}$-sup$(\mathbf{A})$ + $\mathbf{s}$-sup$(\mathbf{B}) - \varepsilon$ is not an a strong upper bound for $\mathbf{C}$ for any $\varepsilon > 0$. Hence by the Proposition 1.3.1 one obtain: $\mathbf{s}$-sup$(\mathbf{C}) = \mathbf{s}$-sup$(\mathbf{A}) + \mathbf{s}$-sup$(\mathbf{B})$.

By using **Theorem 1.3.6.3.A** one obtain:

Theorem 1.3.6.3.B. Let $\mathbf{A}$ and $\mathbf{B}$ be nonempty subsets of ${}^*\mathbb{R}_{\mathbf{d}}$ and $\mathbf{C} = \{a + b : a \in \mathbf{A}, b \in \mathbf{B}\}$.

(**1**) If **A** and **B** are bounded or hyperbounded from above,

hence $\sup(\mathbf{A})$ and $\sup(\mathbf{B})$ exist, then $\sup(\mathbf{C})$ exist and

$$\sup(\mathbf{C}) = \sup(\mathbf{A}) + \sup($$

(**2**) If **A** and **B** are bounded or hyperbounded from below,

hence $\inf(\mathbf{A})$ and $\inf(\mathbf{B})$ exist, then $\inf(\mathbf{C})$ exist and

$$\inf(\mathbf{C}) = \inf(\mathbf{A}) + \inf(\mathbf{B})$$

Theorem 1.3.6.3.C. Setting (**1**). Suppose that **S** is a non-empty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded or hyperbounded from above and $\mathbf{s}$-$\sup \mathbf{S}$ exist

and $\xi \in {}^*\mathbb{R}, \xi > 0$.Then

$$\mathbf{s}\text{-}\sup_{x \in \mathbf{S}} \{\xi \times x\} = \xi \times \left(\mathbf{s}\text{-}\sup_{x \in \mathbf{S}} \{x\}\right) = \xi \times (\mathbf{s}\text{-}\sup \mathbf{S}). \quad (1.3.6$$

Setting (**2**).Suppose that **S** is a non-empty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded or hyperbounded from above and $o\mathbf{s}$-$\sup \mathbf{S}$ exist and $\xi \in {}^*\mathbb{R}$,

$\xi > 0$.Then

$$o\mathbf{s}\text{-}\sup_{x \in \mathbf{S}} \{\xi \times x\} = \xi \times \left(o\mathbf{s}\text{-}\sup_{x \in \mathbf{S}} \{x\}\right) = \xi \times (o\mathbf{s}\text{-}\sup \mathbf{S}). \quad (1.3.6.26.)$$

Proof. (**1**) Let $B = \mathbf{s}\text{-}\sup \mathbf{S}$.Then B is the smallest number such that, for any $x \in \mathbf{S}$,$x \leq B$.Let $\mathbf{T} = \{\xi \times x | x \in \mathbf{S}\}$ .Since $\xi > 0, \xi \times x \leq \xi \times B$ for any $x \in \mathbf{S}$.Hence **T** is bounded or hyperbounded above by $\xi \times B$.By the Theorem 1 and setting (1), **T** has a strong supremum $C, C = \mathbf{s}\text{-}\sup \mathbf{T}$.

Now we have to pruve that $C = \xi \times B$.Since $\xi \times B$ is an apper bound for **T** and C is the smollest apper bound for $\mathbf{T}$,$C \leq \xi \times B$.Now we repeat the

argument above with the roles of **S** and **T** reversed. We know that C is the smallest number such that, for any $y \in \mathbf{T}$,$y \leq C$.Since $\xi > 0$ it follows that $\xi^{-1} \times y \leq \xi^{-1} \times C$ for any $y \in \mathbf{T}$.But $\mathbf{S} = \left\{\xi^{-1} \times y | y \in \mathbf{T}\right\}$ .Hence $\xi^{-1} \times C$ is an apper bound for **S**.But B is a strong supremum for **S**.Hence $B \leq \xi^{-1} \times C$ and $\xi \times B \leq C$.We have shown that $C \leq \xi \times B$ and also that $\xi \times B \leq C$.Thus $\xi \times B = C$.

Theorem 1.3.6.3.D. Let **A** and **B** be nonempty subsets of ${}^*\mathbb{R}_{\mathbf{d}}$ such that $0 \leq \mathbf{A}$,$0 \leq \mathbf{B}$ and $\mathbf{C} = \{a \times b : a \in \mathbf{A}, b \in \mathbf{B}\}$.

(**1.a**) If **A** and **B** are bounded or hyperbounded from above,hence $\mathbf{s}\text{-}\sup(\mathbf{A})$

and $\mathbf{s}\text{-}\sup(\mathbf{B})$ exist, then $\mathbf{s}\text{-}\sup(\mathbf{C})$ exist and

$$\mathbf{s}\text{-}\sup(\mathbf{C}) = [\mathbf{s}\text{-}\sup(\mathbf{A})] \times [\mathbf{s}\text{-}\sup(\mathbf{B})]$$

Proposition 1.3.6.5. Let **A** and **B** be nonempty subsets of ${}^*\mathbb{R}_{\mathbf{d}}$.

(**i**) If for every $a \in \mathbf{A}$ there exists $b \in \mathbf{B}$ with $a \leq b$ and **B** is bounded

from above,then so is **A** and $\sup(\mathbf{A}) \leq \sup(\mathbf{B})$.

(**ii**) If for every $b \in \mathbf{B}$ there exists $a \in \mathbf{A}$ with $a \leq b$ and $\mathbf{A}$ is bounded

from below,then so is $\mathbf{B}$ and $\inf(\mathbf{A}) \leq \inf(\mathbf{B})$.

Proof. (**ii**) Suppose that for every $b \in \mathbf{B}$ there exists $a \in \mathbf{A}$ with $a \leq$ b

and $\mathbf{A}$ is bounded from below. Then $\inf(\mathbf{A})$ exists. For every $b \in \mathbf{B}$ there

is a $\in \mathbf{A}$ such that $a \leq b$. So $\inf(\mathbf{A}) \leq a \leq b$.Therefore $\inf(\mathbf{A})$ is a lower

bound for $\mathbf{B}$.Hence $\mathbf{B}$ is bounded from below and $\inf(\mathbf{B})$ exists. By definition of the infimum (greatest lower bound) one obtain: $\inf(\mathbf{B}) \geq \inf(\mathbf{A})$.

Lemma 1.3.6.1. (**a**) **s-**$\inf({}^*\mathbb{Z})$ and **s-**$\sup({}^*\mathbb{Z})$ is not exists in ${}^*\mathbb{R}_{\mathbf{d}}$.
(**b**) **s-**$\sup({}^*\mathbb{N})$ is not exists in ${}^*\mathbb{R}_{\mathbf{d}}$.
(**c**) ${}^*\mathbb{Z}$ is bounded neither from below nor from above in ${}^*\mathbb{R}_{\mathbf{d}}$.
(**d**) ${}^*\mathbb{N}$ is not bounded from above in ${}^*\mathbb{R}_{\mathbf{d}}$.

Proof. (**a**) Assume that **s-**$\sup({}^*\mathbb{Z})$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$.Then **s-**$\sup({}^*\mathbb{Z}) - 1$ is not an upper bound and hence there exists $n \in {}^*\mathbb{Z}$ such that **s-**$\sup({}^*\mathbb{Z}) - 1 < n$ hence **s-**$\sup({}^*\mathbb{Z}) < n+1$.But since $n+1 \in {}^*\mathbb{Z}$ this is a contradiction. Therefore **s-**$\sup({}^*\mathbb{Z})$ is not exists in ${}^*\mathbb{R}_{\mathbf{d}}$.
(**d**) Assume that ${}^*\mathbb{N}$ has an upper bound, call it $\mathbf{\Theta}$.Hence $\mathbf{\Theta}^{-1}$ is a lower bound for the set $\left\{\mathbf{n}^{-1}\right\}_{\mathbf{n}\in{}^*\mathbb{N}_\infty}$ and consequently $\inf\left(\left\{\mathbf{n}^{-1}\right\}_{\mathbf{n}\in{}^*\mathbb{N}_\infty}\right) \geq \mathbf{\Theta}^{-1} \neq 0$.But we know that $*\text{-}\lim_{\mathbf{n}\to{}^*\infty} \mathbf{n}^{-1} = 0$ which is a contradiction.

Theorem 1.3.6.4. (Generalyzed Archimedean Property of ${}^*\mathbb{R}_{\mathbf{d}}$). For any $\varepsilon \in {}^*\mathbb{R}_{\mathbf{d}}, \varepsilon \approx 0, \varepsilon > 0$ there exists $\mathbf{n} \in {}^*\mathbb{N}$ such that $\mathbf{n}^{-1} < \varepsilon$.

Proof. Since ${}^*\mathbb{N}$ is not hyperbounded from above ε^{-1} is not an upper

bound for ${}^*\mathbb{N}$. Hence there exists $\mathbf{n} \in {}^*\mathbb{N}_\infty$ such that $\mathbf{n} > \varepsilon^{-1}$ and consequently $\mathbf{n}^{-1} < \varepsilon$.

Theorem 1.3.6.5. For every $x \in {}^*\mathbb{R}_{\mathbf{d}}$ such that for the set $\{n \in {}^*\mathbb{Z} | n \leq x\}$one of the next conditions is satisfied:
(**i**) strong supremum **s-**$\sup(\{n \in {}^*\mathbb{Z} | n \leq x\})$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$ or
(**ii**) almost strong supremum **os-**$\sup(\{n \in {}^*\mathbb{Z} | n \leq x\})$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$ or
(**iii**) weak supremum w-$\sup(\{n \in {}^*\mathbb{Z} | n \leq x\})$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$
there exists a unique $m \in {}^*\mathbb{Z}$ such that $m \leq x < m+1$.

Proof. Let $x \in {}^*\mathbb{R}_{\mathbf{d}}$.
Existence: Since ${}^*\mathbb{Z}$ is not hyperbounded from below x is not a lower bound for ${}^*\mathbb{Z}$ hence the set $\mathbf{A} = \{n \in {}^*\mathbb{Z} | n \leq x\}$is not empty. Moreover, x is an upper bound for $\mathbf{A}$ by definition of $\mathbf{A}$.Hence, as a subset of ${}^*\mathbb{R}_{\mathbf{d}}$, $\mathbf{A}$ has a supremum $\sup(\{n \in {}^*\mathbb{Z} | n \leq x\})$,call it $\Delta(x)$. $\Delta(x) - 1$ is not an upper bound for $\mathbf{A}$ hence there exists $m \in \mathbf{A} \subset {}^*\mathbb{Z}$ such that $\Delta(x) - 1 < m$ and consequently $\Delta(x) < m+1$.So $m+1 \notin \mathbf{A}$,so $m+1$.Therefore $m \leq x < m+1$.
Uniqueness: Suppose that $m' \leq x < m'+1$ for $m' \in {}^*\mathbb{Z}$.If $m' < m$, then

$m' + 1 \leq m$ implying $m' \leq x < m' + 1 \leq m \leq x$, a contradiction. $m < m'$ leads to a similar contradiction. So $m = m'$.

Let $E = \left\{x \in {}^*\mathbb{R}_{\mathbf{d}} | x^2 = x \times_{{}^*\mathbb{R}_{\mathbf{d}}} x <_{{}^*\mathbb{R}_{\mathbf{d}}} 2\right\}$.Note that $1^2 = 1 < 2$, so that $1 \in E$ and in particular E is non-empty. Further if $x > 2$ then:
$x^2 = x \times_{{}^*\mathbb{R}_{\mathbf{d}}} x > 2x > 4 > 2$.

Hence 2 is an upper bound for E and so we may define $\zeta \triangleq \sup E$.

Theorem 1.3.6.6. Suppose that $\zeta = \mathbf{s}\text{-}\sup E$ exist.There exists a unique positive number $\zeta \triangleq Ext\text{-}\sqrt{2} \triangleq \#\text{-}\sqrt{2} \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\zeta^2 = \zeta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \zeta =_{{}^*\mathbb{R}_{\mathbf{d}}} 2$.

Proof. Note further that $\zeta > 1 > 0$ is positive. We split the remainder of the proof into showing that $\zeta^2 < 2$ and $\zeta^2 > 2$ both lead to contradictions.

Suppose for a contradiction that $\zeta^2 < 2$.Let $h = \dfrac{1}{2} \min\left(\zeta, \dfrac{2 - \zeta^2}{3\zeta}\right) > 0$.

Then $(\zeta + h)^2 = \zeta^2 + 2h \times \zeta + h^2 < \zeta^2 + 3h \times \zeta < \zeta^2 + (2 - \zeta^2) = 2$.Since $h < \zeta$

and $h < \dfrac{2 - \zeta^2}{3\zeta}$.Hence $\zeta + h \in E$ and since $\zeta = \mathbf{s}\text{-}\sup E$ we get $\zeta + h < \zeta$, a contradiction.

Suppose instead that $\zeta^2 > 2$. Let $h = \dfrac{1}{2}\left(\dfrac{\zeta^2 - 2}{2\zeta}\right) > 0$. As $\zeta - h < \zeta$ there exists $\epsilon \in E$ with $\zeta - h < \epsilon$ by the Strong Approximation Property; then $(\zeta - h)^2 < \epsilon^2 < 2 \Rightarrow \zeta^2 - 2h \times \zeta + h^2 < 2$.

As $h^2 > 0$ this gives $\zeta^2 - 2h \times \zeta < 2$, and so, since $\zeta > 0$, we have $h > \left(\zeta^2 - 2\right)/2\zeta$ which contradicts our choice of h.Finally, by trichotomy, $\zeta^2 = 2$ follows as the only remaining possibility.

Let $E_{<} = \left\{x \in {}^*\mathbb{Q} | x^2 = x \times_{{}^*\mathbb{R}} x <_{{}^*\mathbb{R}} 2\right\}$ and
$E_{>} = \left\{x \in {}^*\mathbb{Q} | x^2 = x \times_{{}^*\mathbb{R}} x >_{{}^*\mathbb{R}} 2\right\}$.

Hence a *Dedekind hyperreal* $\#\text{-}\sqrt{2} \in {}^*\mathbb{R}_{\mathbf{d}}$ is a pair $(U, V) \in \mathbf{P}({}^*\mathbb{Q}) \times \mathbf{P}({}^*\mathbb{Q})$ where $U = E_{<}, V = E_{>}$.

Theorem 1.3.6.7. Let $a \in {}^*\mathbb{R}$ be any positive hyperreal number. Then for any
$n \in {}^*\mathbb{N}$ there exists a unique Dedekind hyperreal number $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ (denoted by$(\sqrt[n]{a})_{\mathbf{d}}$) such that $\alpha^n = a$.

Theorem 1.3.6.8.(**$*$-Density of ${}^*\mathbb{Q}$ in ${}^*\mathbb{R}_{\mathbf{d}}$**). Let $x \in {}^*\mathbb{R}_{\mathbf{d}}$ be a Dedekind

hyperreal number such that $x \pm \varepsilon \neq x$ for any $\varepsilon > 0$, $\varepsilon \approx 0$.. For every $\epsilon > 0$
there exists a hyperrational number $r \in {}^*\mathbb{Q}$ such that $x - \epsilon < r < x + \epsilon$.

Proof. Let $\epsilon > 0$ be given. By the Generalyzed Archimedean Property of

${}^*\mathbb{R}_{\mathbf{d}}$
we can pick $n \in {}^*\mathbb{N}$ with $n^{-1} < \epsilon$.Let $q = [|nx|] \in {}^*\mathbb{N}$.Since $q \leq nx < q + 1$,we
have $\dfrac{q}{n} \leq x < \dfrac{q}{n} + \dfrac{1}{n} < \dfrac{q}{n} + \epsilon$.Now let $r = \dfrac{q}{n} \in {}^*\mathbb{Q}$.Then $r \leq x < r + \epsilon$ and

hence $x - \epsilon < r < x + \epsilon$.

26 Rearrangements of countable infinite series.

Definition 1.3.6.19.(**i**) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$. such that: (**a**) $\forall n\,(\mathbf{s}_n \geq 0)$ or (**b**) $\forall n\,(\mathbf{s}_n < 0)$ or
(**c**) $\{\mathbf{s}_n\}_{n=1}^{\infty} = \{\mathbf{s}_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{\mathbf{s}_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1 \left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0]$,
$\forall n_2 \left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$.

Then external $\flat$-sum of the countable sequence $\mathbf{s}_n$ denoted $\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*\mathbf{s}_n)\right)$

is

$$(\mathbf{a}) \qquad \forall n\,(\mathbf{s}_n \geq 0):$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*\mathbf{s}_n)\right)^{\flat} \triangleq \sup_{k\in\mathbb{N}}\left\{\sum_{n\leq k}({}^*\mathbf{s}_n)^{\#}\right\},$$

$$(\mathbf{b}) \qquad \forall n\,(\mathbf{s}_n < 0):$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*\mathbf{s}_n)\right)^{\flat} \triangleq \inf_{k\in\mathbb{N}}\left\{\sum_{n\leq k}({}^*\mathbf{s}_n)^{\#}\right\}. \tag{1.3.6.24}$$

$$(\mathbf{c}) \quad \forall n_1 \left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0],$$

$$\forall n_2 \left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2 :$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*\mathbf{s}_n)\right)^{\flat} \triangleq \left(\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1}({}^*\mathbf{s}_{n_1})\right)^{\flat} + \left(\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2}({}^*\mathbf{s}_{n_2})\right)^{\flat}.$$

Definition 1.3.6.20.(**i**) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to {}^*\mathbb{R}$
such that: (**a**) $\forall n\,(\mathbf{s}_n \geq 0)$ or (**b**) $\forall n\,(\mathbf{s}_n < 0)$ or
(**c**) $\{\mathbf{s}_n\}_{n=1}^{\infty} = \{\mathbf{s}_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{\mathbf{s}_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1 \left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0]$,
$\forall n_2 \left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$.

Then external $\flat$-sum of the countable sequence $\mathbf{s}_n$ denoted $\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#}\right)^{\flat}$

is

$$(\mathbf{a}) \qquad \forall n\,(\mathbf{s}_n \geq 0):$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#}\right)^{\flat} \triangleq \sup_{k\in\mathbb{N}}\left\{\sum_{n\leq k} s_n^{\#}\right\},$$

$$(\mathbf{b}) \qquad \forall n\,(\mathbf{s}_n < 0):$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#}\right)^{\flat} \triangleq \inf_{k\in\mathbb{N}}\left\{\sum_{n\leq k} s_n^{\#}\right\}. \tag{1.3.6.24$'$}$$

$$(\mathbf{c}) \quad \forall n_1\,(n_1 \in \mathbb{N}_1)\,[\mathbf{s}_{n_1} \geq 0],$$

$$\forall n_2\,(n_2 \in \mathbb{N}_2)\,[\mathbf{s}_{n_2} < 0],\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2:$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#}\right)^{\flat} \triangleq \left(\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1}\mathbf{s}_{n_1}^{\#}\right)^{\flat} + \left(\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2}\mathbf{s}_{n_2}^{\#}\right)^{\flat}.$$

(**ii**) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$,
such that (**a**) $\forall n\,(\mathbf{s}_n \geq 0)$ or (**b**) $\forall n\,(\mathbf{s}_n < 0)$ or
(**c**) $\{\mathbf{s}_n\}_{n=1}^{\infty} = \{\mathbf{s}_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{\mathbf{s}_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0]$,
$\forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$.

Then external $\flat$-sum of the countable sequence $\mathbf{s}_n$ denoted $\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}\mathbf{s}_n\right.$

is

$$(\mathbf{a}) \qquad \forall n\,(\mathbf{s}_n \geq 0):$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n\right)^{\flat} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k} \mathbf{s}_n\right\},$$

$$(\mathbf{b}) \qquad \forall n\,(\mathbf{s}_n < 0):$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n\right)^{\flat} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k} \mathbf{s}_n\right\}. \tag{1.3.6.24$''$}$$

$$(\mathbf{c}) \quad \forall n_1 \left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0],$$

$$\forall n_2 \left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2:$$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n\right)^{\flat} \triangleq \left(\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} \mathbf{s}_{n_1}\right)^{\flat} + \left(\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} \mathbf{s}_{n_2}\right)^{\flat}.$$

Theorem 1.3.6.9.(**i**) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n\,(n \in \mathbb{N})\,[\mathbf{s}_n \geq 0]$, $\sum_{n=1}^{\infty} \mathbf{s}_n = \eta < \infty$, i.e. infinite series $\sum_{n=1}^{\infty} \mathbf{s}_n$ converges to η in $\mathbb{R}$.

Then
$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} ({}^*\mathbf{s}_n)\right)^{\flat} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k} ({}^*\mathbf{s}_n)^{\#}\right\} = ({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \tag{1.3.6.25.a}$$

(**ii**) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n\,(n \in \mathbb{N})\,[\mathbf{s}_n < 0]$, $\sum_{n=1}^{\infty} \mathbf{s}_n = \eta < \infty$, i.e. infinite series $\sum_{n=1}^{\infty} \mathbf{s}_n$ converges to η in $\mathbb{R}$.

Then
$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} ({}^*\mathbf{s}_n)\right)^{\flat} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k} ({}^*\mathbf{s}_n)^{\#}\right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \tag{1.3.6.26.b}$$

(**iii**) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$ such that

(1) $\{\mathbf{s}_n\}_{n=1}^{\infty} = \{\mathbf{s}_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{\mathbf{s}_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1 \left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0]$,

$\forall n_2 \left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$ and

(2) $\displaystyle\sum_{n_1\in\widehat{\mathbb{N}}_1} \mathbf{s}_{n_1} = \eta_1 < \infty, \sum_{n_2\in\widehat{\mathbb{N}}_2} \mathbf{s}_{n_2} = \eta_2 > -\infty.$

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*\mathbf{s}_n)\right)^{\flat}\triangleq$$

Then
$$\left(\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1}({}^*\mathbf{s}_{n_1})\right)^{\flat}+\left(\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2}({}^*\mathbf{s}_{n_2})\right)^{\flat}=({}^*\eta_1)^{\#}-\varepsilon_{\mathbf{d}}+({}^*\eta_2)^{\#}+\varepsilon_{\mathbf{d}}=$$

$$=({}^*\eta_1)^{\#}+({}^*\eta_2)^{\#}-\varepsilon_{\mathbf{d}}\in{}^*\mathbb{R}_{\mathbf{d}}.$$

Theorem 1.3.6.10. Let be $\{a_n\}_{n=1}^{\infty}$ countable sequence $a_n:\mathbb{N}\to{}^*\mathbb{R}_{\mathbf{d}}$,

such that $\forall n\,(a_n\geq 0)$ and $\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}a_n\right)^{\flat}$ external sum of the countable

sequence $\{a_n\}_{n=1}^{\infty}$ denoted by $\mathbf{s}$.

Let be $\{b_n\}_{n=1}^{\infty}$ countable sequence where $b_n=a_{m(n)}$ any rearrangement of

terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

Then external sum $\boldsymbol{\sigma}=\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}b_n\right)^{\flat}$ of the countable sequence $\{b_n\}_{m=1}^{\infty}$

has the same value $\mathbf{s}$ as external sum of the countable sequence $\{a_n\}$,i.e.

$$\boldsymbol{\sigma}=\mathbf{s}. \tag{1.3.6.25}$$

Proof. Let be $\boldsymbol{\sigma}_n=b_1+b_2+...+b_n$ the n-th partial sum of the

sequence $\{b_n\}_{n=1}^{\infty}$ and $\mathbf{s}_m=a_1+a_2+...+a_m$ the m-th partial sum of the

sequence $\{a_n\}_{n=1}^{\infty}$.

It is easy to see that for any given n-th partial sum $\boldsymbol{\sigma}_n=b_1+b_2+...+b_n$

there is exist m-th partial sum $\mathbf{s}_{m(n)}=a_1+a_2+...+a_{m(n)}$ such that:

$$\{a_m\}_{m=1}^{m(n)}\supseteqq\{b_i\}_{i=1}^{n}, \tag{1.3.6.26}$$

and there is exist N-th partial sum $\boldsymbol{\sigma}_{N(m)}=b_1+b_2+...+b_n+...+b_{N(m)}$ such that:

$$\{b_j\}_{j=1}^{N(m)}\supseteqq\{a_i\}_{i=1}^{m(n)}, \tag{1.3.6.27}$$

By using setting and Eqs.(1.3.26)-(1.3.27) one obtain inequality

$$\boldsymbol{\sigma}_n\leq\mathbf{s}_{m(n)}$$

By using **Proposition 1.3.6.5.** one obtain $\sup_{n\in\mathbb{N}}\{\boldsymbol{\sigma}_n\} \leq \sup_{n\in\mathbb{N}}\{\mathbf{s}_{m(n)}\} \leq \sup_{n\in\mathbb{N}}\{\boldsymbol{\sigma}_{N(m)}$

Hence $\boldsymbol{\sigma} \leq \mathbf{s} \leq \boldsymbol{\sigma}$ and finally we obtain $\boldsymbol{\sigma} = \mathbf{s}$.

Theorem 1.3.21.
Theorem 1.3.22.(i) Let be $\{a_n\}_{n=1}^{\infty}$ countable sequence $a_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$,

such that $\forall n\,(a_n \geq 0)$ and $\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n$ external sum of the sequence $\{a_n\}_{n=1}^{\infty}$.

Then for any $c \in {}^*\mathbb{R}_+$ the next equality is satisfied:

$$c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n\right) = \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} c^{\#} \times a_n\right) \quad (1.3.6.30)$$

(ii) Let be $\{a_n\}_{n=1}^{\infty}$ countable sequence $a_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$,
such that $\forall n\,(a_n < 0)$ and $\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n$ external sum of the sequence $\{a_n\}_{n=1}^{\infty}$.

Then for any $c \in {}^*\mathbb{R}_+$ the next equality is satisfied: $c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n\right) = \left(\#Ex\right.$

(iii) Let be $\{\mathbf{s}_n\}_{n=1}^{\infty}$ countable sequence $\mathbf{s}_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$ such that
$\{\mathbf{s}_n\}_{n=1}^{\infty} = \{\mathbf{s}_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{\mathbf{s}_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[\mathbf{s}_{n_1} \geq 0]$,
$\forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[\mathbf{s}_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$
Then the next equality is satisfied:

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n = \#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} \mathbf{s}_{n_1} + \#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} \mathbf{s}_{n_2} \quad (1.3.6.30'')$$

Proof.(i) By using **Definition 1.3.20 (ii)** and **Theorem 1.3.1.3** one obtain

$$c \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n\right) = c \times \sup\left\{\sum_{n\leq m} a_n \middle| m \in \mathbb{N}\right\} =$$
$$= \sup\left[c \times \left\{\sum_{n\leq m} a_n \middle| m \in \mathbb{N}\right\}\right] = \sup\left[\left\{c \times \sum_{n\leq m} a_n \middle| m \in \mathbb{N}\right\}\right] = \quad (1.3.6.31)$$
$$\sup\left[\left\{\sum_{n\leq m} c \times a_n \middle| m \in \mathbb{N}\right\}\right] = \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} c \times a_n\right).$$

Theorem 1.3.13. Let be $\{a_n\}_{n=1}^{\infty}$ countable sequence $a_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$, and $\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n$ external sum of the sequence $\{a_n\}_{n=1}^{\infty}$.

Definition 1.3.20. Let be $\{a_n\}_{n=1}^{\infty}$ arbitrary countable Cauchy sequence $a_n : \mathbb{N} \to \mathbb{R}$. The **upper limit in** ${}^*\mathbb{R}_{\mathbf{d}}$ of the countable sequence $\{a_n\}_{n=1}^{\infty}$

denoted ${}^*\mathbb{R}_{\mathbf{d}}\text{-}\overline{\overline{\lim a_n}}$ is

$$ {}^*\mathbb{R}_{\mathbf{d}}\text{-}\overline{\overline{\lim a_n}} = \inf_{m\in\mathbb{N}} \left(\sup_{n\geq m} \left(a_n^{\#}\right) \right). \quad (1$$

The **lower limit in** ${}^*\mathbb{R}_{\mathbf{d}}$ of the countable sequence $\left\{a_n^{\#}\right\}_{n=1}^{\infty}$

denoted ${}^*\mathbb{R}_{\mathbf{d}}\text{-}\underline{\underline{\lim a_n}}$ is

$$ {}^*\mathbb{R}_{\mathbf{d}}\text{-}\underline{\underline{\lim a_n}} = \sup_{m\in\mathbb{N}} \left(\inf_{n\geq m} \left(a_n^{\#}\right) \right). \quad (1$$

Theorem 1.3.14. Suppose that $\lim_{n\to\infty} a_n = \zeta \in \mathbb{R}$. Then

$$ {}^*\mathbb{R}_{\mathbf{d}}\text{-}\overline{\overline{\lim a_n}} = \ ({}^*\zeta $$

$$ {}^*\mathbb{R}_{\mathbf{d}}\text{-}\underline{\underline{\lim a_n}} = \ ({}^*\zeta $$

Definition 1.3.21. Let be $\{b_n\}_{n=1}^{\infty}$ countable sequence $b_n : \mathbb{N} \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} b_n < \infty$, i.e. infinite series $\sum_{n=1}^{\infty} b_n$ converges in $\mathbb{R}$.

The **upper sum in** ${}^*\mathbb{R}_{\mathbf{d}}$ of the infinite series $\sum_{n=1}^{\infty} b_n$ denoted

$$ \overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n}} \qquad (1.3.6.32)$$

is

$$ \overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n}} \triangleq {}^*\mathbb{R}_{\mathbf{d}}\text{-}\overline{\overline{\lim \left(\sum_{i=1}^{n} b_i^{\#}\right)}} = \inf_{m\in\mathbb{N}} \left(\sup_{n\geq m} \left(\sum_{i=1}^{n} b_i^{\#} \right) \right). \qquad (1.3.6.33)$$

The **lower sum in** ${}^*\mathbb{R}_{\mathbf{d}}$ of the infinite series $\sum_{n=1}^{\infty} b_n$ denoted $\underline{\underline{\#Ext - \sum_{n\in\mathbb{N}} b_n}}$

is

$$ \underline{\underline{\#Ext - \sum_{n\in\mathbb{N}} b_n}} \triangleq {}^*\mathbb{R}_{\mathbf{d}}\text{-}\underline{\underline{\lim \left(\sum_{i=1}^{n} b_i\right)}} = \sup_{m\in\mathbb{N}} \left(\inf_{n\geq m} \left(\sum_{i=1}^{n} b_i \right) \right). \qquad (1.3.6.35)$$

Theorem 1.3.13. Suppose that $\lim_{n\to\infty} \sum_{i=1}^{n} b_i = \zeta \in \mathbb{R}$. Then

$$\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n}} = ({}^*\zeta)^{\#} + \varepsilon_{\mathbf{d}},$$

$$\underline{\underline{\#Ext - \sum_{n\in\mathbb{N}} b_n}} = ({}^*\zeta)^{\#} - \varepsilon_{\mathbf{d}}, \tag{1.3.6.36}$$

Definition 1.3.22. Let be $\{a_n\}_{n=1}^{\infty}$ arbitrary countable sequence $a_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$.

The **upper sum** of the countable sequence $\{a_n\}_{n=1}^{\infty}$ denoted $\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n}}$

is
$$\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n}} \triangleq \inf_{m\in\mathbb{N}}\left(\sup_{n\geq m}\left(\sum_{i=1}^{n} a_i\right)\right). \tag{1.3.6.38}$$

The **lower sum** of the countable sequence a_n denoted $\underline{\underline{\#Ext - \sum_{n\in\mathbb{N}} a_n}}$

is
$$\underline{\underline{\#Ext - \sum_{n\in\mathbb{N}} a_n}} \triangleq \sup_{m\in\mathbb{N}}\left(\inf_{n\geq m}\left(\sum_{i=1}^{n} b_i\right)\right). \tag{1.3.6.40}$$

Theorem 1.3.14. Let be $\{a_n\}_{n=1}^{\infty}$ arbitrary countable sequence

$a_n : \mathbb{N} \to {}^*\mathbb{R}$.Then for every b such that $b \in {}^*\mathbb{R}, b > 0$:

$$(b)^{\#} \times \left(\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} (a_n)^{\#}}}\right) =$$

$$(b)^{\#} \times \left(\underline{\underline{\#Ext - \sum_{n\in\mathbb{N}} (a_n)^{\#}}}\right) =$$

Theorem 1.3.15. Suppose that $\lim_{n\to\infty} \sum_{i=1}^{n} b_i = \zeta \in \mathbb{R}$.Then for every b

$$(b)^{\#} \times \left(\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} ({}^{*}a_n)^{\#}}}\right) = \overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}} (b)^{\#} \times ({}^{*}a_n)^{\#}}} =$$

$$= (b)^{\#} \times ({}^{*}\zeta)^{\#} + (b)^{\#} \times \varepsilon_{\mathbf{d}},$$

such that $b \in {}^{*}\mathbb{R}, b > 0$:

$$(b)^{\#} \times \left(\underline{\underline{\#Ext\text{-}\sum_{n\in\mathbb{N}} ({}^{*}a_n)^{\#}}}\right) = \underline{\underline{\#Ext\text{-}\sum_{n\in\mathbb{N}} (b)^{\#} \times ({}^{*}a_n)^{\#}}} =$$

$$= (b)^{\#} \times \left[({}^{*}\zeta)^{\#} - \varepsilon_{\mathbf{d}}\right].$$

27 I.3.7.The construction non-archimedean field ${}^{*}\mathbb{R}^{\omega}_{\mathbf{d}}$ as Dedekind completion of countable non-standard models of $\mathbb{R}$.

Let ${}^{*}\mathbb{R}_{\omega}$ be a countable field which is elementary equivalent, but not isomorphic to $\mathbb{R}$.

Remark.1.3.7.1. The "elementary equivalence" means that an (arithmetic) expression of first order is true in field ${}^{*}\mathbb{R}_{\omega}$ if and only if it is true in field $\mathbb{R}$. Note that any non-standard model of $\mathbb{R}$ contains an element $\boldsymbol{\upsilon} \in {}^{*}\mathbb{R}_{\omega}$ such

that $\boldsymbol{\upsilon} > x$ for each $x \in \mathbb{R}$.

The canonical way to construct a model for ${}^{*}\mathbb{R}_{\omega}$ uses model theory [31]. We simply take as axioms all axioms of $\mathbb{R}$ and additionally the following countable number of axioms: the existence of an element $\boldsymbol{\upsilon}$ with $\boldsymbol{\upsilon} > 1, \boldsymbol{\upsilon} > 2, ..., \boldsymbol{\upsilon} > n, ...$

Each finite subset of this axioms is satisfied by the standard $\mathbb{R}$.By the compactness theorem in first order model theory, there exists a model which also satisfies the given infinite set of axioms. By the theorem of Löwenheim-Skolem, we can choose such models of countable cardinality.

Each non-standard model ${}^{*}\mathbb{R}$ contains the (externally defined) subset ${}^{*}\mathbb{R}_{\mathbf{fin}}$

Every element $x \in {}^{*}\mathbb{R}_{\mathbf{fin}}$ defines a Dedekind cut: $\mathbb{R} = \{y \in \mathbb{R} |\ y \leq x\} \cup \{y \in \mathbb{R} | y > x\}$.

We therefore get a order preserving map $\mathbf{j}_p$:${}^{*}\mathbb{R}_{\mathbf{fin}} \to \mathbb{R}$ which restricts to

the standard inclusion of the standard irrationals and which respects addition and multiplication. An element of ${}^*\mathbb{R}_{\mathbf{fin}}$ is called infinitesimal,if it is mapped to 0 under the map $\mathbf{j}_p$.

Proposition [30].**1.3.7.1.**Choose an arbitrary subset $M \subset \mathbb{R}$.Then

(**i**) there is a model ${}^*\mathbb{R}^M$ such that $\mathbf{j}_p\left({}^*\mathbb{R}^M_{\mathbf{fin}}\right) \supset M$.

(**ii**) the cardinality of ${}^*\mathbb{Q}^M$ can be chosen to coincide with **card** (M),if M is infinite.

Proof. Choose $M \subset \mathbb{R}$. For each $m \in M$ choose $q_1^m < q_2^m < ... < ... < p_2^m < p_1^m$
with $\lim_{k\to\infty} q_k^m = \lim_{k\to\infty} p_k^m = m$.
We add to the axioms of $\mathbb{R}$ the following axioms:$\forall m \in M\ \exists e_m$ such that $q_k^m < e_m < p_k^m$ for all $k \in \mathbb{N}$.
Again, the standard $\mathbb{R}$ is a model for each finite subset of these axioms, so that the compactness theorem implies the existence of ${}^*\mathbb{R}^M$ as required, where the cardinality of ${}^*\mathbb{R}^M$ can be chosen to be the cardinality of the set of axioms, i.e. of M, if M is infinite. Note that by construction $\mathbf{j}_p(e_m) = e_m$.

Remark.**1.4.2.2.3.**It follows in particular that for each countable subset of $\mathbb{R}$ we can find a countable model of ${}^*\mathbb{R}$ such that the image of fp contains this subset. Note, on the other hand, that the image will only be countable, so that the different models will have very different ranges.

Definition 1.4.2.2.1.[30]. A Cauchy sequence in ${}^*\mathbb{R}_\omega$ is a sequence $(a_k)_{k\in\mathbb{N}}$ such that for every $\varepsilon \in {}^*\mathbb{R}_\omega$, $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that:
$\forall m_{m\,>\,n_\varepsilon} \forall n_{n\,>\,n_\varepsilon}\left[|\ a_m - a_n| < \varepsilon\right]$.

Definition 1.4.2.2.2. We define Cauchy completion ${}^*\mathbb{R}^\omega_{\mathbf{c}} \triangleq \left[{}^*\mathbb{R}_\omega\right]_{\mathbf{c}}$ in the canonical way as equivalence classes of Cauchy sequences.

Remark.**1.4.2.2.4.** This is a standard construction and works for all ordered fields.The result is again a field, extending the original field. Note that, in our case,each point in ${}^*\mathbb{R}^\omega_{\mathbf{c}}$ is infinitesimally close to a point in ${}^*\mathbb{R}$.

Remark.**1.4.2.2.5.** In many non-standard models of $\mathbb{R}$, there are no countable sequences $(a_k)_{k\in\mathbb{N}}$ tending to zero which are not eventually zero.

Proposition [30].**1.3.7.2.** Assume that ${}^*\mathbb{R}$ is countable.

28 I.4.The construction non-archimedean field ${}^*\mathbb{R}_{\mathbf{c}}$.

29 I.4.1.Completion of ordered group and fields in general by using 'Cauchy pregaps'.

We cketch here the aspects of the general theory that is concerned with completion ordered group and fields, to be constructed by using 'Cauchy pregaps' [32].

Throughout in this section we shall only consider fields which are associative, linear algebra over ground field $\Bbbk$, where $\Bbbk = \mathbb{Q}, \mathbb{R}, {}^*\mathbb{Q}, {}^*\mathbb{R}$.

30 I.4.1.1.Totally ordered group and fields

Definition 1.4.1.1.1. Let $(K, +, \cdot)$ be a field and let $(\circ \leq \circ)$ be a binary relation on K.Then $(K, +, \cdot, \leq)$ is an ordered field if

(**i**) $(K, \leq)$ is totally ordered set,

(**ii**) $(K, +, \leq)$ is an ordered group and

(**iii**) $a, b \in K^+ \implies a \cdot b \in K^+$.

Note the standard convention that the order $\leq$ on an ordered field K is necessarily a total order.

Let K be an ordered field. It is easy to see from Definition 1.4.1.1. that $|a| \cdot |b| = |a \cdot b|$.

Let K, L be an ordered fields. An imbeding of K in L is an algebra monomorphism from K into L which is isotonic. A surjective embedding is an isomorphism. In the case where exist an isomorphism K onto L then K and L are isomorphic, and we write $K \cong L$.

Definition 1.4.1.1.2. Let A be an algebra. Then:

1. $A[X]$ denotes the algebra of polinomials $p(X)$ with coefficients in A;

2. ${}^*A[X]$ denotes the algebra of hyperpolinomials $P(X)$ with coefficients in ${}^*A$;

3. in the case where A is a subalgebra of an algebra B and $b \in B$, $A[b] \triangleq \{p(b) \,|p \in A[X]\}$;

4. in the case where A is a subalgebra of an algebra B and $b \in {}^*B, {}^*A[b] \triangleq \{P(b) \,|P \in {}^*A[X]\}$;

Definition 1.4.1.1.3. Let A be a subalgebra of an algebra B and $b \in B$. Then:

1. $b \in B$ is algebraic over A if there exist $p \in A[X] \setminus \{0\}$ with $p(b) = 0$;

2. $b \in B$ is transcendental over A if is not algebraic over A;

3. $b \in {}^*B$ is hyperalgebraic over A if there exist $P \in {}^*A[X] \setminus \{0\}$ with $P({}^*b) = 0$;

4. $b \in {}^*B$ is hypertranscendental over A if is not hyperalgebraic over A;

Definition 1.4.1.1.4. Let A be a subalgebra of an algebra B and $b \in B$. Then:

1.$b \in B$ is w**-transcendental over** A if:

(a) $b \in B$ is transcendental over A and

(b) there exist $P \in {}^*A[X] \setminus \{0\}$ with $P({}^*b) = 0$
 or with $P({}^*b) \approx 0$;

2. $b \in B$ is #**-transcendental over** A if:

(a) $b \in B$ is transcendental over A and

(b) there is no exist $P \in {}^*A[X] \setminus \{0\}$ with $P({}^*b) \approx 0$.

Example 1.4.1.1.1. Number $\pi \in \mathbb{R}$ is w-transcendental over $\mathbb{Q}$. There exist

$P_\pi(X) \in {}^*\mathbb{Q}[X] \setminus \{0\}$ with $P_\pi({}^*\pi) \approx 0$ where

$$P_\pi(X) = [\sin(X)]_{N \in {}^*\mathbb{N}_\infty} = \left({}^* \sum_{m=}^{N \in} \right.$$

Example 1.4.1.1.2. Number $\ln 2 \in \mathbb{R}$ is w-transcendental over $\mathbb{Q}$. There exist

$P_{\ln 2}(X) \in {}^*\mathbb{Q}[X] \setminus \{0\}$ with $P_{\ln 2}(\ln 2) - 2 \approx 0$ where

$$P_{\ln 2}(X) = [\exp(X)]_{N}$$

Definition 1.4.1.1.4. Let A be a subalgebra of an algebra B.Then:

(**i**) The algebra B is an algebraic extension of A if each
$b \in B$ is algebraic over A;otherwise B is transcendental extension of A.

(**ii**) The algebra ${}^*B$ is an hyperalgebraic extension of A if each
$b \in {}^*B$ is hyperalgebraic over A;otherwise B is hypertranscendental extension of A.

(**iii**) The algebra ${}^*B$ is an w-transcendental extension of A if each
$b \in {}^*B$ is w-transcendental over A.

(**iv**) The algebra ${}^*B$ is an #-transcendental extension of A if each
$b \in {}^*B$ is #-transcendental over A.

Definition 1.4.1.1.5. Let K be a field.

(**i**) A field K is algebraically closed iff there is no
field L which is a proper algebraic extension of K,or,equivalently,
K is algebraically closed iff each non constant $p \in K[X]$ has a root in K.

(**ii**) A field K is hyperalgebraically closed iff there is no
field L which is a proper hyperalgebraic extension of K,or,equivalently,
K is hyperalgebraically closed iff each non constant $P \in {}^*K[X]$ has a root in K, i.e. $P({}^*b) = 0$ fore some $b \in K$.

Definition 1.4.1.1.6. An ordered field K is real-closed if

(a) it has no proper algebraic extension to an ordered field,
or, equivalently,if

(b) the complexification $K_{\mathbb{C}}$ of K is algebraically closed,
or, equivalently,if

(c) every positive element in K is a square and every polinomial over K of odd degree has a root in K.

Let K be a real-closed ordered field and take some $c \in K^{+} \setminus \{0\}$ and $n \in \mathbb{N}$. There is a anique element $b \in K^{+} \setminus \{0\}$ such that $b^n = c$, and so there is a map $\psi : \mathbb{Q}_{+} \to K^{+} \setminus \{0\}$ where $\psi : \alpha \longmapsto c^{\alpha}$. Thus

$$\psi(0) = 1, \psi(1) = c, \qquad (1.4.1.1.3)$$

$$\psi(\alpha + \beta) = \psi(\alpha) \cdot \psi(\beta), (\alpha, \beta \in \mathbb{Q}_{+}).$$

31 I.4.1.2.Cauchy completion of ordered group and fields.

Definition 1.4.1.2.1. Let $\langle A, B \rangle$ be a pregap in totally ordered group G. Then $\langle A, B \rangle$ is a Cauchy pregap if A has no maximum,B has no minimum, and,for each $\varepsilon > 0, \varepsilon \in G$,there exist $a \in A$ and $b \in B$ with $b < a + \varepsilon$.
Definition 1.4.1.2.2. The group G is Cauchy complete if, for each Cauchy pregap there exists $x \in G$ with $a << x << b$.
Remark.1.4.1.1. Thus totally ordered group G is Cauchy complete iff there are no Cauchy gaps.
Remark.1.4.1.2. The element x arising in the above definition is necessarily unique.
Example 1.4.1.1. (i) The group $(\mathbb{R}, +)$ is certainly Cauchy complete.
(ii) The group $({}^{*}\mathbb{R}, +)$ is Cauchy complete.
(iii) The monoid $({}^{*}\mathbb{R}_{\mathbf{d}}, +)$ is certainly Cauchy complete.
Definition 1.4.1.3. The set of the all Cauchy pregaps in totally ordered group G we denote by $C(G)$.
Definition 1.4.1.4. A totally ordered group G is discrete if the set $G^{+} \setminus \{0\}$ is empty or has a minimum element and is non-discrete otherwise.

For any $\langle A_1, B_1 \rangle \in C(G)$ and $\langle A_2, B_2 \rangle \in Cp(G)$ we hawe $\langle A_1 + A_2, B_1 + B_2 \rangle \in Cp(G)$.Let us define sum of the classes $[\langle A_1, B_1 \rangle] \in Cl[Cp(G)]$ and $[\langle A_2, B_2 \rangle] \in Cl[Cp(G)]$ by formula

$$[\langle A_1, B_1 \rangle] + [\langle A_2, B_2 \rangle]$$

Then $+$ is well defined in $H_G = Cl[Cp(G)]$ and $\{Cl[Cp(G)], +\} =$ $\{H_G, +\}$ is an abelian group.The map

$$\iota : \{G, +\} \hookrightarrow \{H_G,$$

is a canonical group morphism. It is easy to see that $\{H_G, +\}$ is a totally ordered group. Let $\left\langle \hat{A}, \hat{B} \right\rangle$ be a Cauchy pregap in H_G,and

define

$$\mathbf{A}=\bigcup\left\{A_\alpha|\left[(A_\alpha,B_\alpha)\right]\in\hat{A}\text{ for some }B_\alpha\right\}$$

$$\mathbf{B}=\bigcup\left\{B_\alpha|\left[(A_\alpha,B_\alpha)\right]\in\hat{B}\text{ for some }A_\alpha\right\},$$

hence $\langle\mathbf{A},\mathbf{B}\rangle\in Cp\left(G\right)$ and $\hat{A}<<[(\mathbf{A},\mathbf{B})]<<\hat{B}$ and $\{H_G,+\}$ is Cauchy complete. On it we have the following result:

Theorem 1.4.1.1.[32].Let G be a totally ordered non-discrete group. Then the group $\{H_G,+\}$ is defined by formula (1.4.1.1) is a Cauchy completion of G.

Definition 1.4.1.5. An ordered field K is Cauchy complete if the totally ordered group $\{K,+\}$ is Cauchy complete.

Theorem 1.4.1.2.[32].Let K be an ordered field.Then the Cauchy completion $\widetilde{K}$ of the group $\{K,+\}$ can be made into an ordered field in such a way that K is a subfield of $\widetilde{K}$. If K is real-closed, then so is $\widetilde{K}$.

Proof. Suppose $a_1,a_2\in\widetilde{K},a_1>0,a_2>0$ and $a_1=[(A_1,B_1)]$,

$a_2=[(A_2,B_2)]$ .We may suppose that $A_1,A_2\subset K^+\backslash\{0\}$ .Set

$$A =$$

$$B =$$

Then $\langle A,B\rangle\in Cp\left(\{K,+\}\right)$ .Define

$$a_1\cdot a_2=[\langle A_1,B_1\rangle]\cdot[\langle A_2,B_2$$

The operation $(\circ\cdot\circ)$ is well defined in $\widetilde{K}$ and $a_1\cdot a_2>0$.If $a_1<0$, $a_2>0$ set $a_1\cdot a_2=-\left((-a_1)\cdot a_2\right)$, etc.

It is simple to check that the group $\left\{\widetilde{K},+\right\}$ together with product $(\circ\cdot\circ)$ is an ordered field $\widetilde{K}\triangleq\left\{\widetilde{K},+,\cdot\right\}$ with the required properties.

Let K' be any ordered field containing K as an order-dense subfield. Then there is an isotonic morphism from K' into $\widetilde{K}$, and so $\widetilde{K}$ is the maximum ordered field containing K as an proper order-dense subfield. On it we have the following result:

Theorem 1.4.1.3.[32].Let K be an ordered field.Then K is Cauchy complete iff no ordered field L containing K as an proper order-dense subfield.

Standard main tool for anderstanding structure of the totally ordered external group well be Hahn's embedding theorem. We shall associate to a totally ordered external and internal group $(G,+,\leq)$ a 'value set' $\Gamma^{\#}$,define a

collections $F\left(\mathbb{R},\Gamma\right),F\left({}^{*}\mathbb{R},\Gamma^{\#}\right)$ of 'formal power series' over Γ and $\Gamma^{\#}$ and imbed

G into $F\left(\mathbb{R},\Gamma\right)$ or $F\left({}^{*}\mathbb{R},\Gamma^{\#}\right)$ correspondingly.

Let us define the value set of external group G.

Definition 1.4.1.6.[32]. Let $(G,+,\leq)$ be a totally ordered external group, i.e. $(G,+,\leq)\in V^{Ext}$ and let $x,y\in G$.Set:

(i) $x = o(y)$ iff $\forall n_{n\in\mathbb{N}}\,[n \cdot |x| \leq |y|]$;
(ii) $x = O(y)$ iff $\exists m_{m\in\mathbb{N}}\,[|x| \leq m \cdot |y|]$;

(iii) $x \,\tilde{}\, y$ iff $[x = O(y)] \wedge [y = O(x)]$.
For each $y \in G$ the sets $\{x | x = o(y)\}$ and $\{x | x = O(y)\}$ are absolutely convex subsets of G.Is clear that $(\circ \,\tilde{}\, \circ)$ is an equivalence relation on G. Each $\tilde{}$ -equivalence class (other than $\{0\}$) is the union of an interval contained in $G^{+} \setminus \{0\}$ and an interval contained in $G^{-} \setminus \{0\}$.
Definition 1.4.1.7. Let $(G, +, \leq)$ be a totally ordered external group. The set $\Gamma = \Gamma_G = (G \setminus \{0\}) /^{\sim}$ of equivalence classes in the value set of G and the elements of Γ are the archimedian classes of G.
The quotient map from $G \setminus \{0\}$ onto Γ is denoted by $v : G \setminus \{0\} \to \Gamma$.It is the archimedian valuation on G.Set $v(x) \leq v(y)$ for $x, y \in G \setminus \{0\}$ iff $y = O(x)$.
It is easy checked that $\leq$ is well defined on Γ,hence $(\Gamma, \leq)$ is a totally

ordered set, such that

$$v(x+y) \geq \min\{v(x), v(y)\},$$

$$v(x+y) = \min\{v(x), v(y)\} \text{ if } [v(x) \neq v(y)] \vee [(x \in G^{+})$$

Definition 1.4.1.8. Let $\left(\breve{G}, +, \leq\right)$ a totally ordered internal group,i.e. $\left(\breve{G}, +, \leq\right) \in V^{Int}$ and let $x, y \in G$.In particular $\breve{G} = {}^{*}(G, +, \leq)$ for some $(G, +, \leq)$,i.e. in particular $\breve{G}$ is an standard group.
Set:

32 I.4.2.1.The construction non-archimedean field ${}^{*}\mathbb{R}_{\mathbf{c}}$ by using Cauchy hypersequence in ancountable field ${}^{*}\mathbb{Q}$.

Let ${}^{*}\mathbb{Q}_{\omega_\alpha} \triangleq {}^{*}\mathbb{Q}, \omega < \omega_\alpha$ be a ancountable field which is elementary equivalent,
to $\mathbb{Q}$.The "elementary equivalence" means that an (arithmetic) expression of
first order is true in field ${}^{*}\mathbb{Q}_{\omega_\alpha}$ if and only if it is true in field $\mathbb{Q}$.Note that any
non-standard model of $\mathbb{Q}$ contains an element $\mathbf{e} \in {}^{*}\mathbb{Q}_{\omega}$ such that $\mathbf{e} > q$ for each $q \in \mathbb{Q}$.
We define Cauchy completion $[{}^{*}\mathbb{Q}_{\omega_\alpha}]_{\mathbf{c}} \triangleq {}^{*}\mathbb{R}_{\mathbf{c}}$ in the canonical way as equivalence classes of Cauchy hypersequences.
Remark.1.4.2.1.1. This is a general construction and works for all nonstandard

ordered fields ${}^*\mathbb{k}$.The result is again a field $[{}^*\mathbb{k}]_{\mathbf{c}}$ which is potentially different from extending the original field $\mathbb{k}$, and we actually see that ${}^*\mathbb{R}_{\mathbf{c}}$ is different from ${}^*\mathbb{Q}$ as a consequence of Theorem

Remark.1.4.2.1.2. In many non-standard ancountable models of $\mathbb{Q}$, there are no countable sequences tending to zero which are not eventually zero.Thus dealing with analysis over field ${}^*\mathbb{R}$ we are compelled to enter into consideration hypersequences of various classes: $\mathbf{s}_{\mathbf{n}\in{}^*\mathbb{N}} : {}^*\mathbb{N} \to {}^*\mathbb{Q}, \mathbf{s}_{\mathbf{n}\in{}^*\mathbb{N}} : {}^*\mathbb{N} \to {}^*\mathbb{R}$ and $\mathbf{s}_{\mathbf{n}\in{}^*\mathbb{N}} : {}^*\mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$

Definition 1.4.2.1.1. A hypersequence $\mathbf{s}_{\mathbf{n}} : {}^*\mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}} \supset {}^*\mathbb{R} \supset {}^*\mathbb{Q}, \mathbf{n} \in {}^*\mathbb{N}$ tends to a $*$-limit α ($\alpha \in^* \mathbb{Q},^* \mathbb{R}$ or ${}^*\mathbb{R}_{\mathbf{d}}$) in ${}^*\mathbb{Q},^*\mathbb{R}$ or ${}^*\mathbb{R}_{\mathbf{d}}$ iff

$$\exists \alpha \left(\alpha \in {}^*\mathbb{R}_{\mathbf{d}}\right) \forall \varepsilon_{\varepsilon>0} \left(\varepsilon \in {}^*\mathbb{R}_{\mathbf{d}}\right) \exists \mathbf{n}_0 \left(\mathbf{n}_0 \in {}^*\mathbb{N}_\infty\right) \forall \mathbf{n} [$$

We write $*\text{-}\lim_{\mathbf{n} \longrightarrow {}^*\infty} \mathbf{s}_{\mathbf{n}_{\mathbf{n}\in{}^*\mathbb{N}}} = \alpha$ or $\underset{\mathbf{n} \longrightarrow {}^*\infty}{*\text{-}\lim} \mathbf{s}_{\mathbf{n}} = \alpha$ iff condition (1.4.1.1.1) is satisfied.

Definition 1.4.2.1.2. A hypersequence $\mathbf{s}_{\mathbf{n}} : {}^*\mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}} \supseteqq {}^*\mathbb{Q}$ is divergent in ${}^*\mathbb{R}_{\mathbf{d}}$, or tends to ${}^*\infty$ iff

$$\forall r_{r>0} \left(r \in {}^*\mathbb{R}\right) \exists \mathbf{n}_0 \left(\mathbf{n}_0 \in {}^*\mathbb{N}_\infty\right) \forall \mathbf{n} \left[\mathbf{n} \geqslant \mathbf{n}_0 \implies |\mathbf{s}_{\mathbf{n}}| > r\right]. \quad (1.4.$$

Lemma 1.4.2.1.1. Suppose that $\mathbf{s}_{\mathbf{n}} : {}^*\mathbb{N} \to {}^*\mathbb{R} \supset {}^*\mathbb{Q}, \mathbf{n} \in {}^*\mathbb{N}$.

(a) If $\underset{\mathbf{n} \longrightarrow {}^*\infty}{*\text{-}\lim} \mathbf{s}_{\mathbf{n}}$ exists in ${}^*\mathbb{R}$, then it is unique.

(b) If $\underset{\mathbf{n} \longrightarrow {}^*\infty}{*\text{-}\lim} \mathbf{s}_{\mathbf{n}}$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$, then it is unique.

That is if $*\text{-}\lim_{\mathbf{n} \longrightarrow {}^*\infty} \mathbf{s}_{\mathbf{n}} = \alpha_1, * - \lim_{\mathbf{n} \longrightarrow {}^*\infty} \mathbf{s}_{\mathbf{n}} = \alpha_2$ then $\alpha_1 = \alpha_2$.

Proof. (a) Let ε be any positive number $\varepsilon > 0, \varepsilon \in {}^*\mathbb{R} \supset {}^*\mathbb{Q}$.Then, by definition, we must be able to find a number N_1 so that $|\mathbf{s}_{\mathbf{n}} - \alpha_1| < \varepsilon$ whenever $\mathbf{n} \geq N_1$. We must also be able to find a number N_2 so that $|\mathbf{s}_{\mathbf{n}} - \alpha_2| < \varepsilon$ whenever $\mathbf{n} \geq N_2$. Take $\mathbf{m}$ to be the maximum of N_1 and N_2. Then both assertions $|\mathbf{s}_{\mathbf{m}} - \alpha_1| < \varepsilon$ and $|\mathbf{s}_{\mathbf{m}} - \alpha_2| < \varepsilon$ are true.
This by using triangle inequality allows us to conclude that
$|\alpha_1 - \alpha_2| = |(\alpha_1 - \mathbf{s}_{\mathbf{m}}) + (\mathbf{s}_{\mathbf{m}} - \alpha_2)| \leq |\alpha_1 - \mathbf{s}_{\mathbf{m}}| + |\mathbf{s}_{\mathbf{m}} - \alpha_2| < 2\varepsilon$. So that $|\alpha_1 - \alpha_2| < 2\varepsilon$. But ε can be any positive infinite small number whatsoever. This could only be true if $\alpha_1 = \alpha_2$, which is what we wished to show.

(b)

Definition 1.4.2.1.3. A *Cauchy hypersequence in* ${}^*\mathbb{Q},^*\mathbb{R}$ and ${}^*\mathbb{R}_{\mathbf{d}}$ is a sequence $\mathbf{s}_{\mathbf{n}} : {}^*\mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}} \supseteqq {}^*\mathbb{Q}$ with the following property: for every $\varepsilon \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\varepsilon > 0$,there exists an $\mathbf{n}_0 \in {}^*\mathbb{N}_\infty$ such that $\mathbf{m}, \mathbf{n} \geqslant n_0$ implies $|\mathbf{s}_{\mathbf{m}} - \mathbf{s}_{\mathbf{n}}| < \varepsilon$,

i.e.

$$\forall \varepsilon_{(\varepsilon\in{}^*\mathbb{R}_{\mathbf{d}})} \left(\varepsilon > 0\right) \exists \mathbf{n}_0 \left(\mathbf{n}_0 \in {}^*\mathbb{N}_\infty\right) \left[\mathbf{m}, \mathbf{n} \geqslant n_0 \implies |\mathbf{s}_{\mathbf{m}} - \mathbf{s}_{\mathbf{n}}| < \varepsilon\right]$$

Lemma 1.4.2.1.2. A hypersequence of numbers ${}^*\mathbb{Q},{}^*\mathbb{R}$ and ${}^*\mathbb{R}_{\mathbf{d}}$, that converges is Cauchy hypersequence.
Lemma 1.4.2.1.3. A Cauchy hypersequence $(\mathbf{s_n})_{\mathbf{n}\in{}^*\mathbb{N}}$ in ${}^*\mathbb{Q}$ and ${}^*\mathbb{R}$,is bounded or hyperbounded.
Proof. Choose in (1.4.2.1) $\varepsilon = 1$. Since the sequence $(\mathbf{s_n})_{\mathbf{n}\in{}^*\mathbb{N}}$ is Cauchy, there exists a positive hyperinteger $N \in {}^*\mathbb{N}$ such that $|\mathbf{s_i} - \mathbf{s_j}| < 1$whenever $\mathbf{i},\mathbf{j} \geq N$.In particular,$|\mathbf{s_i} - \mathbf{s}_N| < 1$ whenever $\mathbf{i} \geq N$ By the triangle inequality,

$|\mathbf{s_i}| - |\mathbf{s}_N| \leq |\mathbf{s_i} - \mathbf{s}_N|$ and therefore,$|\mathbf{s_i}| < |\mathbf{s}_N| + 1$ for all $\mathbf{i} \geq N$.

Definition 1.4.2.1.4. Cauchy hypersequences $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ and $(y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$,can be
added, multiplied and compared as follows:
(**a**) $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} + (y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} = (x_\mathbf{n} + y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$,
(**b**) $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \times (y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} = (x_\mathbf{n} \times y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$,
(**c**) $\dfrac{(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}}{(y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}} = \left(\dfrac{x_\mathbf{n}}{y_\mathbf{n}}\right)_{\mathbf{n}\in{}^*\mathbb{N}}$ iff $\forall\mathbf{n}\,(\mathbf{n}\in{}^*\mathbb{N})\,[y_n \neq 0]$,
(**d**) $(x_\mathbf{n})^{-1}_{\mathbf{n}\in{}^*\mathbb{N}} = (x^{-1}_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ iff $\forall\mathbf{n}\in{}^*\mathbb{N}(y_\mathbf{n}\neq 0)$,
(**e**) $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \geq (y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ if and only if for every $\epsilon > 0, \epsilon \in {}^*\mathbb{Q}$ there exists
an integer $\mathbf{n}_0$ such that $x_\mathbf{n} \geq y_\mathbf{n} - \epsilon$ for all $\mathbf{n} > \mathbf{n}_0$.

Definition 1.4.2.1.5. Two Cauchy hypersequences $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ and $(y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ are
called equivalent: $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (y_n)_{\mathbf{n}\in{}^*\mathbb{N}}$ if the hypersequence

$(x_\mathbf{n} - y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$has $*$-limit zero,i.e. $*\text{-}\lim_{\mathbf{n}\to{}^*\infty}(x_\mathbf{n} - y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} = 0$.
Lemma 1.4.2.1.4. If $(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (x'_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ and $(y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (y'_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$, are two
pairs of equivalent Cauchy hypersequences, then:

(**a**) hypersequence $(x_\mathbf{n}+y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ is Cauchy and

$$(x_\mathbf{n} + y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (x'_\mathbf{n} + y'_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$$

(**b**) hypersequence $(x_\mathbf{n} - y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ is Cauchy and

$$(x_\mathbf{n} - y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (x'_\mathbf{n} - y'_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}, \qquad (1.4.2.1.5)$$

(**c**) hypersequence $(x_\mathbf{n}\times y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ is Cauchy and

$$(x_\mathbf{n} \times y_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (x'_\mathbf{n} \times y'_\mathbf{n})_{\mathbf{n}\in{}^*}$$

(**d**) hypersequence $\left(\dfrac{x_\mathbf{n}}{y_\mathbf{n}}\right)_{\mathbf{n}\in{}^*\mathbb{N}}$ is Cauchy and

$$\left(\frac{x_\mathbf{n}}{y_\mathbf{n}}\right)_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} \left(\frac{x'_\mathbf{n}}{y'_\mathbf{n}}\right)_{\mathbf{n}\in{}^*\mathbb{N}}$$

iff $\forall\mathbf{n}_{(\mathbf{n}\in{}^*\mathbb{N})}\,[(y_\mathbf{n} \not\equiv 0) \wedge (y'_\mathbf{n} \not\equiv 0) \wedge (y_\mathbf{n} \not\approx_\mathbf{c} 0)]$,
(**e**) hypersequence $(x_\mathbf{n}+0_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}$ where $\forall\mathbf{n}_{(\mathbf{n}\in{}^*\mathbb{N})}\,[0_\mathbf{n} = 0]$
is Cauchy and

$$(x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} + (0_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}} \approx_\mathbf{c} (x_\mathbf{n})_{\mathbf{n}\in{}^*\mathbb{N}}, \qquad (1.4.2.1.8)$$

here $(0_{\mathbf{n}})_{\mathbf{n}\in\ {}^*\mathbb{N}}$ is a *null hypersequence,*

(f) hypersequence $(x_{\mathbf{n}} \times 1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ where $\forall \mathbf{n}_{(\mathbf{n}\in {}^*\mathbb{N})} [1_{\mathbf{n}} = 1]$ is Cauchy and

$$(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}, \quad (1.4.2.1.9)$$

here $(1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ is a unit hypersequence.

(g) hypersequence $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (x_{\mathbf{n}})^{-1}_{\mathbf{n}\in {}^*\mathbb{N}}$ is Cauchy and

$$(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (x_{\mathbf{n}})^{-1}_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} ($$

iff $\forall \mathbf{n}_{(\mathbf{n}\in {}^*\mathbb{N})} \left[(x_{\mathbf{n}} \not\equiv 0) \wedge \left(x_{\mathbf{n}} \not\approx_{\mathbf{c}} (0_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \right) \right].$

Proof. **(a)** From definition of the Cauchy hypersequences one obtain:

$$\exists \varepsilon_1 \exists \mathbf{m}_{(\mathbf{m}\in {}^*\mathbb{N}_\infty)} \forall \mathbf{k} \, (\mathbf{k} \geqslant \mathbf{m}) \, \forall \mathbf{l} \, (\mathbf{l} \geqslant \mathbf{m}) \, [(|x_{\mathbf{k}} - x_{\mathbf{l}}| < \varepsilon_1) \wedge (|y_{\mathbf{k}} - y_{\mathbf{l}}| < \varepsilon_1)]. \quad (1.4.2.1.11)$$

Suppose $\varepsilon_1 = \varepsilon/2$, then from formula above we can to choose $\mathbf{m} = \mathbf{m}(\varepsilon_1)$ such that for all $\mathbf{k} \geqslant \mathbf{m}, \mathbf{l} \geqslant \mathbf{m}$ valid the next inequalities:

$$|(x_{\mathbf{k}} + y_{\mathbf{k}}) - (x_{\mathbf{l}} + y_{\mathbf{l}})| = |(x_{\mathbf{k}} -$$

$$\leqslant |(x_{\mathbf{k}} - x_{\mathbf{l}})| + |(y_{\mathbf{k}} - y_{\mathbf{l}})| <$$

$$|(x'_{\mathbf{k}} + y'_{\mathbf{k}}) - (x'_{\mathbf{l}} + y'_{\mathbf{l}})| = |(x'_{\mathbf{k}} -$$

$$\leqslant |(x'_{\mathbf{k}} - x'_{\mathbf{l}})| + |(y'_{\mathbf{k}} - y'_{\mathbf{l}})| <$$

From **Definition 1.4.2.1.5.** and inequalities (1.4.12) we have the statement (**a**).

(b) Similarly proof the statement (a) we have the next inequalities:

$$|(x_{\mathbf{k}} - y_{\mathbf{k}}) - (x_{\mathbf{l}} - y_{\mathbf{l}})| = |(x_{\mathbf{k}} - x_{\mathbf{l}}) + (y_{\mathbf{k}} - y_{\mathbf{l}})| \leqslant$$

$$\leqslant |(x_{\mathbf{k}} - x_{\mathbf{l}})| + |(y_{\mathbf{k}} - y_{\mathbf{l}})| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad (1.4.2.1.13)$$

$$|(x'_{\mathbf{k}} - y'_{\mathbf{k}}) - (x'_{\mathbf{l}} - y'_{\mathbf{l}})| = |(x'_{\mathbf{k}} - x'_{\mathbf{l}}) - (y'_{\mathbf{k}} - y'_{\mathbf{l}})| \leqslant$$

$$\leqslant |(x'_{\mathbf{k}} - x'_{\mathbf{l}})| + |(y'_{\mathbf{k}} - y'_{\mathbf{l}})| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

From **Definition 1.4.2.1.5** and inequalities (1.4.13) we have the statement (**b**).

(c) $\forall \mathbf{k} \, (\mathbf{k} \geqslant \mathbf{m})$ and $\forall \mathbf{l} \, (\mathbf{l} \geqslant \mathbf{m})$ we have the next inequalities:

$$
\begin{gathered}
|x_{\mathbf{k}}\cdot y_{\mathbf{k}}-x_{\mathbf{l}}\cdot y_{\mathbf{l}}|=|(x_{\mathbf{k}}\cdot y_{\mathbf{k}}-x_{\mathbf{l}}\cdot y_{\mathbf{k}})+(x_{\mathbf{l}}\cdot y_{\mathbf{k}}-x_{\mathbf{l}}\cdot y_{\mathbf{l}})|\leqslant\\
\leqslant|x_{\mathbf{k}}-x_{\mathbf{l}}|\cdot|y_{\mathbf{l}}|+|y_{\mathbf{k}}-y_{\mathbf{l}}|\cdot|x_{\mathbf{l}}|\,,\\
|x'_{\mathbf{k}}\cdot y'_{\mathbf{k}}-x'_{\mathbf{l}}\cdot y'_{\mathbf{l}}|=|(x'_{\mathbf{k}}\cdot y'_{\mathbf{k}}-x'_{\mathbf{l}}\cdot y'_{\mathbf{k}})+(x'_{\mathbf{l}}\cdot y'_{\mathbf{k}}-x'_{\mathbf{l}}\cdot y'_{\mathbf{l}})|\leqslant\\
\leqslant|x'_{\mathbf{k}}-x'_{\mathbf{l}}|\cdot|y'_{\mathbf{l}}|+|y'_{\mathbf{k}}-y'_{\mathbf{l}}|\cdot|x'_{\mathbf{l}}|\,,\\
|x_{\mathbf{k}}\cdot y_{\mathbf{k}}-x'_{\mathbf{k}}\cdot y'_{\mathbf{k}}|=|(x_{\mathbf{k}}\cdot y_{\mathbf{k}}-x_{\mathbf{k}}\cdot y'_{\mathbf{k}})+(x_{\mathbf{k}}\cdot y'_{\mathbf{k}}-x'_{\mathbf{k}}\cdot y'_{\mathbf{k}})|\leqslant\\
\leqslant|x_{\mathbf{k}}-x'_{\mathbf{k}}|\cdot|y'_{\mathbf{k}}|+|y_{\mathbf{k}}-y'_{\mathbf{k}}|\cdot|x_{\mathbf{k}}|\,.
\end{gathered}
\tag{1.4.2.1.14}
$$

From definition Cauchy hypersequences one obtain $\exists c\forall\mathbf{k}:|x_{\mathbf{k}}|\leqslant c,|y_{\mathbf{k}}|\leqslant c$, $|x'_{\mathbf{k}}|\leqslant c,|y'_{\mathbf{k}}|\leqslant c$. From **Definition 1.4.2.1.5** and inequalities (1.4.14) we have the statement (**c**).

Let $\Re^*_{\mathbf{c}}$ denote the set of the all equivalence classes $\left\{(x_n)_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\in\Re^*_{\mathbf{c}}$

Using Lemma 1.4.1. one can define an equivalence relation $\approx_{\mathbf{c}}$,which is

compatible with the operations defined above, and the set ${}^*\mathbb{R}_{\mathbf{c}}=\Re^*_{\mathbf{c}}/\approx_{\mathbf{c}}$

is satisfy of the all usual field axioms of the hyperreal numbers.

Lemma 1.4.2.1.5. Suppose that $\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\},\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\},\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\in{}^*\mathbb{R}_{\mathbf{c}}$, then:

$$
\begin{array}{ll}
(\mathbf{a}) & \left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}=\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}, \\
(\mathbf{b}) & \left[\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right]+\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}= \\
 & =\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left[\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right], \\
(\mathbf{c}) & \left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left[\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right]= \\
 & =\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}, \\
(\mathbf{d}) & \left[\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right]\times\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}= \\
 & =\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left[\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right], \\
(\mathbf{e}) & \left[\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right]\times\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}= \\
 & =\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left[\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\right], \\
(\mathbf{f}) & \left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}+\left\{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}=\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}, \\
(\mathbf{g}) & \left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\cdot\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}^{-1}=\left\{(1_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}, \\
(\mathbf{i}) & \left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}=\left\{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}, \\
(\mathbf{j}) & \left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\{(1_{\mathbf{n}})\}=\{(x_{\mathbf{n}})\}, \\
(\mathbf{k}) & \left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}<\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\wedge\left\{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}<\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\implies \\
 & \implies\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}<\left\{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}\times\left\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\right\}.
\end{array}
\tag{1.4.2.1.15}
$$

Proof. Statements **(a),(b),(c),(d),(e),(f),(g),(i) (j)** and **(k)** is evidently from

Lemma.1.4.1 and definition of the equivalence relation $\approx_{\mathbf{c}}$.

33 I.4.2.2.The construction non-archimedean field ${}^*\mathbb{R}^{\omega}_{\mathbf{c}}$ as Cauchy completion of countable non-standard models of $\mathbb{Q}$.

Let ${}^*\mathbb{Q}_{\omega}$ be a countable field which is elementary equivalent, but not isomorphic to $\mathbb{Q}$.

Remark.1.4.2.2.1. The "elementary equivalence" means that an (arithmetic)

expression of first order is true in field ${}^*\mathbb{Q}_\omega$ if and only if it is true in field $\mathbb{Q}$.

Note that any non-standard model of $\mathbb{Q}$ contains an element $\mathbf{e} \in {}^*\mathbb{Q}_\omega$ such

that $\mathbf{e} > q$ for each $q \in \mathbb{Q}$.

The canonical way to construct a model for ${}^*\mathbb{Q}_\omega$ uses model theory [30],[31]. We simply take as axioms all axioms of $\mathbb{Q}$ and additionally the following countable number of axioms: the existence of an element $\mathbf{e}$ with $\mathbf{e} > 1, \mathbf{e} > 2, ..., \mathbf{e} > n, ...$.Each finite subset of this axioms is satisfied by the standard $\mathbb{Q}$.By the compactness theorem in first order model theory, there exists a model which also satisfies the given infinite set of axioms. By the theorem of Löwenheim-Skolem, we can choose such models of countable cardinality.

Each non-standard model ${}^*\mathbb{Q}$ contains the (externally defined) subset ${}^*\mathbb{Q}_{\mathbf{fin}}$

Every element $x \in {}^*\mathbb{Q}_{\mathbf{fin}}$ defines a Dedekind cut: $\mathbb{Q} = \{q \in \mathbb{Q} |\ q \leq x\} \cup \{q \in \mathbb{Q} | q > x\}$.

We therefore get a order preserving map $\mathbf{j}_{op}$:${}^*\mathbb{Q}_{\mathbf{fin}} \to \mathbb{R}$ which restricts to the standard inclusion of the standard rationals and which respects addition and multiplication. An element of ${}^*\mathbb{Q}_{\mathbf{fin}}$ is called infinitesimal,if it is mapped to 0 under the map $\mathbf{j}_{op}$.

Proposition [30].**1.4.2.2.1.**Choose an arbitrary subset $M \subset \mathbb{R}$.Then
(**i**) there is a model ${}^*\mathbb{Q}^M$ such that $\mathbf{j}_{op}\left({}^*\mathbb{Q}^M_{\mathbf{fin}}\right) \supset M$.
(**ii**) the cardinality of ${}^*\mathbb{Q}^M$ can be chosen to coincide with **card** (M),if M is infinite.

Proof. Choose $M \subset \mathbb{R}$. For each $m \in M$ choose $q_1^m < q_2^m < ... < ... < p_2^m < p_1^m$ with $\lim_{k\to\infty} q_k^m = \lim_{k\to\infty} p_k^m = m$.

We add to the axioms of $\mathbb{Q}$ the following axioms:$\forall m \in M\ \exists e_m$ such that $q_k^m < e_m < p_k^m$ for all $k \in \mathbb{N}$.

Again, the standard $\mathbb{R}$ is a model for each finite subset of these axioms, so that the compactness theorem implies the existence of ${}^*\mathbb{Q}^M$ as required, where the cardinality of ${}^*\mathbb{Q}^M$ can be chosen to be the cardinality of the set of axioms, i.e. of M, if M is infinite. Note that by construction $\mathbf{j}_{op}(e_m) = e_m$.

Remark.1.4.2.2.2. It follows in particular that for each countable subset of $\mathbb{R}$ we can find a countable model ${}^*\mathbb{Q}_\omega$ of ${}^*\mathbb{Q}$ such that the image of $\mathbf{j}_{op}(\circ)$ contains this subset.Note, on the other hand, that the image will only be countable, so that the different models will have very different ranges.

Definition 1.4.2.2.1.[30]. A Cauchy sequence in ${}^*\mathbb{Q}_\omega$ is a sequence $(a_k)_{k\in\mathbb{N}}$ such that for every $\varepsilon \in {}^*\mathbb{Q}_\omega$, $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that:

$\forall m_{m\,>\,n_\varepsilon} \forall n_{n\,>\,n_\varepsilon} \left[|\, a_m - a_n| < \varepsilon \right].$

Definition 1.4.2.2.2. We define Cauchy completion ${}^*\mathbb{R}^{\omega}_{\mathbf{c}} \triangleq [{}^*\mathbb{Q}_\omega]_{\mathbf{c}}$ in the canonical way as equivalence classes of Cauchy sequences.

Remark.1.4.2.2.3. This is a standard construction and works for all ordered fields.The result is again a field, extending the original field. Note that, in our case,each point in $[{}^*\mathbb{Q}_\omega]_{\mathbf{c}}$ is infinitesimally close to a point in ${}^*\mathbb{Q}$.

Remark.1.4.2.2.4. In many non-standard models of $\mathbb{Q}$, there are no countable sequences $(a_k)_{k\in\mathbb{N}}$ tending to zero which are not eventually zero.

34 II.Euler's proofs by using non-archimedean analysis on the pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ revisited. original proof of the Goldbach-Euler Theorem revisited.

That's what he's infected me with,he thought. His madness.That's why I've come here.That's what I want here. A strange and very new feeling overwhelmed him. He was aware that the feeling was really not new at all, that it had been hidden in him for a long time, but that he was acknowledging it only now, and everything was falling into place.

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"Roadside Picnic"

Euler's paper of 1737 "Variae Observationes Circa Series Infinitas," is Euler's first paper that closely follows the modern Theorem-Proof format. There are no definitions in the paper, or it would probably follow the Definition-Theorem-Proof format. After an introductory paragraph in which Euler tells part of the story of the problem, Euler gives us a theorem and a "proof". Euler's "proof" begins with an 18-th century step that treats *infinity as a number.* Such steps became unpopular among rigorous mathematicians about a hundred years later. He takes x to be the "sum" of the harmonic series:

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + ... + \frac{1}{n} + ... \qquad (2.4.0)$$

The Euler's original proofs is one of those examples of *completely misuse* of divergent series to obtain *completely correct results* so frequent during the seventeenth and eighteenth centuries.The acceptance of Euler's proofs seems to lie in the fact that,at the time,Euler (and most of his contemporaries) actually

manipulated a model of real numbers which included infinitely large and infinitely small numbers. A model that much later Bolzano would try to build on solid grounds and that today is called “nonstandard” after A.Robinson definitely established it in the 1960’s [1],[2],[3],[4],[5]. This last approach, though, is completely in tune with Euler’s proof [7] Nevertheless using ideas borrowed from modern nonstandard analysis the same reconstruction rigorous by modern Robinsonian standards *is not found.* In particular ”nonstandard” proof proposed in paper [7] is not completely nonstandard becourse authors use the solution Catalan’s conjecture [9]

Unfortunately completely correct proofs of the Goldbach-Euler Theorem, was presented many authors as rational reconstruction only in terms which could be considered rigorous by modern Weierstrassian standards.

In this last section we show how, a few simple ideas from non-archimedean analysis on the pseudoring ${}^{*}\mathbb{R}_{\mathbf{d}}$, vindicate Euler’s work.

Theorem 2.1.1. (Euler [6],[8]) Consider the following series, infinitely continued,

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+...$$

whose denominators, increased by one, are all the numbers which are powers of the integers, either squares or any other higher degree.Thus each term may be expressed by the formula

$$\frac{1}{m^n-1}$$

where m and n are integers greater than one. The sum of this series is 1.

Proof. Let

$$\mathbf{h}=\mathbf{cl}\left(1+\frac{1}{2},1+\frac{1}{2}+\frac{1}{3},1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},...\right) \tag{2.4.3}$$

$$1=\mathbf{cl}\left(\frac{1}{2},\frac{1}{2}+\frac{1}{4},\frac{1}{2}+\frac{1}{4}+\frac{1}{8},\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16},\right.$$

$$\left.\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32},...,\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...+\frac{1}{2^i},...\right)-\varepsilon_1,$$

from Eq.(2.4.3), as we have

$$\varepsilon_1\approx 0,$$

$$\varepsilon_1=\mathbf{cl}\left(\frac{1}{2^M},\frac{1}{2^{M+1}},...,\frac{1}{2^{M+i}},...\right)$$

we obtain

$$\mathbf{h}-1=\mathbf{cl}\left(1,1+\frac{1}{3},1+\frac{1}{3}+\frac{1}{5},1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6},1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7},\right. \tag{1.4.5}$$

$$\left.1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10},...\right)-\varepsilon_1$$

$$\frac{1}{2}=\mathbf{cl}\left(\frac{1}{3},\frac{1}{3}+\frac{1}{9},\frac{1}{3}+\frac{1}{9}+\frac{1}{27},...,\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...+\frac{1}{3^i},...\right)-\varepsilon_2,$$

from Eq.(2.4.5), as we have

$$\varepsilon_2\approx 0,$$

$$\varepsilon_2=\frac{1}{2}\mathbf{cl}\left(\frac{1}{3^M},\frac{1}{3^{M+1}},...,\frac{1}{3^{M+i}},...\right)$$

we obtain

$$\mathbf{h}-\left(1+\frac{1}{2}\right)=\mathbf{cl}\left(1,1+\frac{1}{5},1+\frac{1}{5}+\frac{1}{6},1+\frac{1}{5}+\frac{1}{6}+\frac{1}{7},\right. \tag{2.4.7}$$

$$\left.1+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11},...\right)-(\varepsilon_1+\varepsilon_2).$$

$$\frac{1}{4}=\mathbf{cl}\left(\frac{1}{5},\frac{1}{5}+\frac{1}{25},\frac{1}{5}+\frac{1}{25}+\frac{1}{125},...,\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+...+\frac{1}{5^i},...\right)-\varepsilon_3,$$

from Eq.(2.4.7), as we have

$$\varepsilon_3\approx 0,$$

$$\varepsilon_3=\frac{1}{4}\mathbf{cl}\left(\frac{1}{5^M},\frac{1}{5^{M+1}},...,\frac{1}{5^{M+i}},...\right)$$

we obtain

$$\mathbf{h}-\left(1+\frac{1}{2}+\frac{1}{4}\right)= \tag{2.4.9}$$

$$\mathbf{cl}\left(1+\frac{1}{6},1+\frac{1}{6}+\frac{1}{7},1+\frac{1}{6}+\frac{1}{7}+\frac{1}{10},...\right)-(\varepsilon_1+\varepsilon_2+\varepsilon_3).$$

Proceeding similarly, i.e. deleting all the all terms that remain,we get

$$\mathbf{h}-[\Im_n]=$$

$$=\mathbf{cl}\left(1+\frac{1}{5},...,1+\frac{1}{m(n')},1+\frac{1}{m(n')}+...,...\right)- \tag{2.4.10}$$

$$-\left(\#Ext-\textstyle\sum_{n\in\mathbb{N}}\varepsilon_n\right),$$

$$m>n'(n)$$

where
$$\Im_n = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, ..., \frac{1}{n'(n)}, ...\right) \qquad (2.4.11)$$
whose denominators, increased by one, are all the numbers

which are not powers. From Eq.(2.4.10) we obtain

$$\mathbf{h} - [\Im_n] =$$
$$1 + \left(\#Ext - \textstyle\sum_{n\in\mathbb{N}} \varepsilon_n\right) =$$
$$= 1 + \epsilon$$

$$\epsilon = \#Ext - \textstyle\sum_{n\in\mathbb{N}} \varepsilon_n \approx 0.$$

Thus we obtain
$$\mathbf{h} - [\Im_n] = 1 + \epsilon, \qquad (2.4.1$$

Substitution Eq.(2.4.3) into Eq.(2.4.13) gives

$$1 + \epsilon = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + .$$

$$\epsilon \approx 0$$

series whose denominators, increased by one, are all the powers of the integers and whose sum is one.

Time passed, and more or less coherent thoughts came to him. Well,that's it,he thought unwillingly. The road is open. He could go down right now, but it was better, of course, to wait a while. The meatgrinders can be tricky.

He got up,automatically brushed off his pants, and started down into the quarry.The sun was broiling hot, red spots floated before his eyes, the air was quivering on the floor of the quarry, and in the shimmer it seemed that the ball was dancing in place like a buoy on the waves. He went past the bucket, superstitiously picking up his feet higher and making sure not to step on the splotches. And then, sinking into the rubble, he dragged himself across the quarry to the dancing, winking ball.

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"Roadside Picnic"

35 III.Non-archimedean analysis on the extended hyperreal line ${}^{*}\mathbb{R}_{\mathbf{d}}$ and transcendence conjectures over field $\mathbb{Q}$.Proof that $e+\pi$ and $e\cdot\pi$ is irrational.

36 III.1.Hyperrational approximation of irrational numbers.

The next simple result shows that in a way the hyperrationals already "incorporate" the real numbers (see e.g. [25] Ch.II Thm. 2)

Theorem 3.1.1. Let ${}^{*}\mathbb{Q}_{\mathbf{fin}}$ be the ring of finite hyperrationals, and let $\Im$ be the maximal ideal of its infinitesimals. Then $\mathbb{R}$ and ${}^{*}\mathbb{Q}_{\mathbf{fin}}/\Im$ are isomorphic as ordered fields.

Theorem 3.1.2.(Standard form of Dirichlet's Approximation Theorem).

Let be $\alpha\in\mathbb{R}$ positive real number and $n\in\mathbb{N}$ a positive integer. Then there is an integer $k\in\mathbb{N}$ and an integer $b\in\mathbb{N}$ with $0<k<n$, for which

$$-\frac{1}{n}<k\cdot\alpha-b<\frac{1}{n}. \qquad (\mathbf{DAP})$$

Definition 3.1.1. A "**D**-approximation" to α is a rational number $\dfrac{p}{q}\in\mathbb{Q}$, whose

denominator is a positive integer $q\in\mathbb{N}$, with $$\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^2}.$$

Theorem 3.1.3. If $\alpha\in\mathbb{R}$ is irrational it has infinitely many **D**-approximations.

Remark 3.1.1.[37]. Let sequence $\dfrac{p_n}{q_n}$ be a convergent to α in the sense that

$$\alpha=\frac{p_n}{q_n}+\frac{\theta_{q_n}}{q_n^2}, \qquad (\mathbf{DAP3})$$

$(p_n,q_n)=1,|\theta_{q_n}|<1,n=0,1,2,...$

i.e. there is exist infinite sequence $(p_n,q_n)\in\mathbb{Z}\times\mathbb{N},n=0,1,2,...$ such that $q_{n+1}>q_n$ and

$$\alpha = \frac{p_n}{q_n} + \frac{\theta_{q_n}}{q_n^2}, \qquad \textbf{(DAP4)}$$

$$(p_n, q_n) = 1, |\theta_{q_n}| < 1.$$

Theorem 3.1.3 shows that each irrational number α has infinitely many convergents of the form **DAP4**.

Definition 3.1.2. (**i**) Let $\alpha \in \mathbb{R}$ is irrational number. A $*$-**D**-approximation to ${}^*\alpha \in {}^*\mathbb{R}$ is a number $\frac{P}{Q} \in {}^*\mathbb{Q}$, $P \in {}^*\mathbb{Z}_\infty$ whose denominator is

a positive hyperinteger $Q \in {}^*\mathbb{N}_\infty$,with
$$\left|{}^*\alpha - \frac{P}{Q}\right| < \frac{1}{Q^2},$$
$$(P, Q) = 1.$$

(**ii**) Let $\alpha \in \mathbb{R}$ is irrational number. A "#-**D**-approximation" to $({}^*\alpha)^\# \in {}^*\mathbb{R}_{\mathbf{d}}$

is a Wattenberg hyperrational number $\frac{P^\#}{Q^\#} \in {}^*\mathbb{Q}_{\mathbf{d}}$, $P^\# \in {}^*\mathbb{Z}_{\infty,\mathbf{d}}$ whose

denominator is a positive hyperinteger $Q^\# \in {}^*\mathbb{N}_{\infty,\mathbf{d}}$,with
$$\left|({}^*\alpha)^\# - \frac{P^\#}{Q^\#}\right| < \frac{1}{Q}$$
$$\left(P^\#, Q^\#\right) = 1^\#.$$

Definition 3.1.3. (**i**) Let $\alpha \in \mathbb{R}$ is irrational number. A hyperrational approximation to α is a $*$-**D**-approximation to ${}^*\alpha \in {}^*\mathbb{R}$.
(**ii**) Let $\alpha \in \mathbb{R}$ is irrational number. A Wattenberg hyperrational approximation to α is a #-**D**-approximation" to $({}^*\alpha)^\# \in {}^*\mathbb{R}_{\mathbf{d}}$.

Theorem 3.1.3.(Nonstandard form of Dirichlet's Approximation Theorem).

(**1**) If α is irrational it has infinitely many $*$-**D**-approximations such that for any

two $*$-**D**-approximations P_1/Q_1 and P_2/Q_2 the next equality is satisfied

$$\frac{P_1}{Q_1} \approx \frac{P_2}{Q_2},$$

i.e.

$$({}^*\alpha)^\# = \left(\frac{P_1}{Q_1}\right)^\# \pmod{\varepsilon_{\mathbf{d}}}.$$

$$^*\alpha = \frac{P}{Q} + \frac{\theta_Q}{Q^2}$$

(**2**) If $\alpha \in \mathbb{R}$ is irrational then $^*\alpha \in {}^*\mathbb{R}$ has representation

$$P \in {}^*\mathbb{Z}_\infty, Q \in {}^*\mathbb{N}_\infty, ($$

$$(P,Q) = 1, |\theta_Q|$$

$$({}^*\alpha)^\# = \frac{P}{Q}$$

(**3**) If $\alpha \in \mathbb{R}$ is irrational then $({}^*\alpha)^\# \in {}^*\mathbb{R}_\mathbf{d}$ has representation

$$P^\# \in {}^*\mathbb{Z}_{\infty,\mathbf{d}}, Q^\# \in$$

$$\left(P^\#, Q^\#\right) = 1$$

Definition 3.1.4. A real number $\alpha \in \mathbb{R}$ is a Liouville number if for every positive integer $m \in \mathbb{N}$, there is exist infinite sequence

$(p_n, q_n) \in \mathbb{Z}\times\mathbb{N}, n = 0, 1, 2, ...$ such that $$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q^m}.$$

Remark 3.1.2. This is well known that all Liouville numbers are transcendental.

From (3.1.1) one obtain directly:

Theorem 3.1.4. (**i**) Any Liouville number $\alpha_l \in \mathbb{R}$ for every positive hyperinteger

$N \in {}^*\mathbb{N}_\infty$ has a hyperrational approximation such that $$0 < \left|{}^*\alpha_l - \frac{P}{Q}\right| < \frac{1}{Q^N},$$

$$P \in {}^*\mathbb{Z}_\infty, N, Q \in {}^*\mathbb{N}_\infty, (P,Q)$$

(**ii**) Any Liouville number $\alpha_l \in \mathbb{R}$ for every positive hyperinteger

$N \in {}^*\mathbb{N}_\infty$ has a Wattenberg hyperrational approximation such that

Theorem 3.1.5. Every Liouville number $\alpha_l \in \mathbb{R}$ are #-transcendental over field $\mathbb{Q}$, i.e., there is no real $\mathbb{Q}$-analytic function

$g_\mathbb{Q}(x) = \sum_{n=0}^{\infty} a_n x^n < \infty, 0 \leq |x| \leq r \leq e$ with rational coefficients

$a_0, a_1, ..., a_n, ... \in \mathbb{Q}$ such that $g_{\mathbb{Q}}(\alpha_l)$.

37 III.2.Proof that e is #-transcendental.

Definition 3.2.1. Let $g(x) : \mathbb{R} \to \mathbb{R}$ be any real analytic function $g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty}$

$\forall n [a_n$

defined on an open interval $I \subset \mathbb{R}$ such that $0 \in I$.

We call this function given by Eq.(3.2.1) $\mathbb{Q}$-analytic function and denote $g_{\mathbb{Q}}(x)$.

Definition 3.2.2. Arbitrary transcendental number $z \in \mathbb{R}$ is called #-transcendental number over field $\mathbb{Q}$, if no exist $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z) = 0$,i.e. for every $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ the inequality $g_{\mathbb{Q}}(z) \neq 0$ is satisfies.

Definition 3.2.3. Arbitrary transcendental number z called w-transcendental

number over field $\mathbb{Q}$,if z is not #-transcendental number over field $\mathbb{Q}$,i.e. exist $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z) = 0$.

Example 3.2.1. Number π is transcendental but number π is not #-transcendental number over field $\mathbb{Q}$ as

(**1**) function $\sin x$ is a $\mathbb{Q}$-analytic and

(**2**) $\sin\left(\frac{\pi}{2}\right) = 1$,i.e.

$$-1 + \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + ... + \frac{(-1)^{2n+1} \pi^{2n+1}}{2^{2n+1}(2n+1)!} + ... = 0. \tag{3.2.2}$$

Theorem 3.2.1. Number e is #-transcendental over field $\mathbb{Q}$.

Proof I. To prove e is #-transcendental number we must show that e it is not w-transcendental, i.e., there is no exist real $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} a_n x^n, 0 \leq |x| \leq r \leq e$ with rational coefficients $a_0, a_1, ..., a_n, ... \in \mathbb{Q}$

such that
$$\sum_{n=0}^{\infty} a_n e^n = 0. \tag{3.2.3}$$

Suppose that e is w-transcendental, i.e., there is exist an $\mathbb{Q}$-analytic

function $\breve{g}_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} \breve{a}_n x^n$,with rational coefficients: $$\breve{a}_0 = \frac{k_0}{m_0}, \breve{a}_1 = \frac{k_1}{m_1}, ..., \breve{a}_n = \frac{k}{n}$$

$$\breve{a}_0 > 0$$

such that the next equality is satisfied: $$\sum_{n=0}^{\infty} \breve{a}_n e^n = 0.$$

Hence there is exist sequences $\{n_i\}_{i=0}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$ such that

$$\forall k \left[\sum_{n=0}^{n_i \leq k} \breve{a}_n e^n > 0\right], \lim_{i\to\infty} \sum_{n=0}^{n_i} \breve{a}_n e^n = 0,$$
$$\forall k \left[\sum_{n=1}^{n_j \leq k} \breve{a}_n e^n < 0\right], \lim_{j\to\infty} \sum_{n=1}^{n_j} \breve{a}_n e^n = 0. \tag{3.2.6}$$

From Eqs.(3.2.6) by using definitions one obtain the next

equalities:

$$({}^*\breve{a}_0)^{\#} + \left[\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}\setminus\{0\}} ({}^*\breve{a}_n)^{\#} \times ({}^*e^n)^{\#}}}\right]_{\varepsilon} =$$
$$= \left[\left({}^*\left(\inf_{i\in\mathbb{N}}\left(\sum_{n=0}^{n_i} \breve{a}_n e^n\right)\right)\right)^{\#} + \varepsilon_{\mathbf{d}}\right]_{\varepsilon} =$$
$$= \left({}^*\left(\inf_{i\in\mathbb{N}}\left(\sum_{n=0}^{n_i} \breve{a}_n e^n\right)\right)\right)^{\#} + \varepsilon^{\#} \times \varepsilon_{\mathbf{d}} = \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}, \tag{3.2.}$$

$$\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}$$

and by similar way

$$(^*\breve{a}_0)^{\#} + \left[\underline{\underline{\#Ext\text{-}\sum_{n\in\mathbb{N}\setminus\{0\}} (^*\breve{a}_n)^{\#} \times (^*e^n)^{\#}}}\right]_{\varepsilon} =$$

$$= \sup_{j\in\mathbb{N}} \left(\sum_{n=1}^{n_j} \breve{a}_n e^n\right) - \varepsilon^{\#} \times \varepsilon_{\mathbf{d}} = -\varepsilon^{\#} \times \varepsilon_{\mathbf{d}}, \tag{3.2.8}$$

$$\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}.$$

Let us considered hypernatural number $\Im \in {}^*\mathbb{N}_\infty$ defined by countable

sequence

$$\Im = (m_0, m_0 \times m_1, ..., m_0 \times m_1 \times ... \times m_n, ...)$$

$$\Im^{\#} \times (^*\breve{a}_0)^{\#} + \Im^{\#} \times \left[\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}\setminus\{0\}} (^*\breve{a}_n)^{\#} \times (}}\right.$$

$$\Im^{\#} \times (^*\breve{a}_0)^{\#} + \left[\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}\setminus\{0\}} \Im^{\#} \times (^*\breve{a}_n)^{\#} \times}}\right.$$

By using Eq.(3.2.7) and Eq.(3.2.9) one obtain

$$= \Im_0^{\#} + \left[\overline{\overline{\#\text{-}Ext\text{-}\sum_{n\in\mathbb{N}\setminus\{0\}} \Im_n^{\#} \times (^*e^n)^{\#}}}\right| \Im^{\#} \times ($$

$$c \in \mathbb{R},$$

$$\Im_n^{\#} \triangleq \Im^{\#} \times (^*\breve{a}_n)^{\#}, n \in$$

Now we have to pruve that Eq.(3.2.10) leads to contradiction.

Proof I.

Part I. Let be

$$M_0(n,p) = \int_0^{+\infty} \left[\frac{x^{p-1}\left[(x-1)\dots(x-n)\right]^p e^{-x}}{(p-1)!}\right] dx \neq 0, \tag{3.2.11}$$

$$M_k\left(n,p\right)=e^k\int\limits_k^{+\infty}\left[\frac{x\left[^{p-1}\left(x-1\right)...\left(x-n\right)\right]^p e^{-x}}{\left(p-1\right)!}\right]dx, \tag{3.2.12}$$

$$k=1,2,...$$

$$\varepsilon_k\left(n,p\right)=e^k\int\limits_0^{k}\left[\frac{x^{p-1}\left[\left(x-1\right)...\left(x-n\right)^p\right]e^{-x}}{\left(p-1\right)!}\right]dx, \tag{3.2.13}$$

$$k=1,2,...$$

where $p\in\mathbb{N}$ this is any prime number.Using Eqs.(3.2.9.)-(3.2.13.) by simple calculation one obtain:

$$M_k\left(n,p\right)+\varepsilon_k\left(n,p\right)=e^kM_0\neq 0, \tag{3.2.14}$$

$$k=1,2,...$$

and consequently

$$e^k=\frac{M_k\left(n,p\right)+\varepsilon_k\left(n,p\right)}{M_0} \tag{3.2.15}$$

$$k=1,2,...$$

By using equality $$x^{p-1}\left[\left(x-1\right)...\left(x-n\right)\right]^p=\left(-1\right)^n\left(n!\right)^n x^{n-1}+\sum_{\mu=p+1}^{(n+1)\times p}c_{\mu-1}x^{\mu-1},$$

$$c_\mu\in\mathbb{Z},\mu=p,p+1,...,\left[\left(n+1\right)\times p\right]-1,$$

$$M_0(n,p) = (-1)^n (n!)^p \frac{\Gamma(p)}{(p-1)!} + \sum_{\mu=p+1}^{(n+1)\times p} c_{\mu-1}$$

$$= (-1)^n (n!)^p + c_p p + c_{n+1} p(p+1) + \ldots$$

from Eq.(3.2.11.) one obtain

$$= (-1)^n (n!)^p + p \times \Theta_1, \Theta_1 \in \mathbb{Z},$$

$$\Gamma(\mu) = \int_0^{+\infty} x^{\mu-1} e^{-x} dx.$$

$$M_0(n,p) = (-1)^n (n!)^p + p \times \Theta_1, \Theta_1 \in$$

Thus

$$M_0(n,p) = (-1)^n (n!)^p + p \cdot \Theta_1(n,p), \Theta_1(n,p) \in \mathbb{Z}.$$

By subsitution $x = k + u \implies dx = du$ from Eq.(3.2.13.) one obtain

$$M_k(n,p) = \int_0^{+\infty} \left[\frac{(u+k)^{p-1} [(u+k-1) \times \ldots \times u \times \ldots \times (u+k-n)]^p e^{-u}}{(p-1)!} \right] du \qquad (3.2.19)$$

$$k = 1, 2, \ldots$$

By using equality

$$(u+k)^{p-1} [(u+k-1) \times \ldots \times u \times \ldots \times (u+k-n)]^p = \sum_{\mu=p+1}^{(n+1)\times p} d_{\mu-1} u^{\mu-1},$$

$$d_\mu \in \mathbb{Z}, \mu = p, p+1, \ldots, [(n+1) \times p] - 1,$$

and by subsitution Eq.(3.2.20) into Eq.(3.2.19) one obtain

$$M_k(n,p) = \frac{1}{(p-1)!} \int_0^{+\infty} \sum_{\mu=p+1}^{(n+1)\times p}$$

$$\Theta_2(n,p$$

$$k = 1, 2$$

There is exists sequences $a(n), n \in \mathbb{N}$ and $g_k(n), k \in \mathbb{N}, n \in \mathbb{N}$

such that

$$|x(x-1)\ldots(x-n)| < a(n), 0 \le x \le n, \qquad (3.2.22)$$

$$\left| x(x-1)\ldots(x-n) e^{-x+k} \right| < g_k(n), 0 \le x \le n, k = 1, 2, \ldots .$$

Substitution inequalities (3.2.22.) into Eq.(3.2.13.) gives

$$\varepsilon_k(n,p) \le g_k(n)\frac{[a(n)]^{p-1}}{(p-1)!}\int_0^k$$

$$^*M_0(\mathbf{n},\mathbf{p}) = {}^*\left(\int_0^{+\infty}\left[\frac{x^p}{}\right.\right.$$

By using transfer, from Eq.(3.2.11.) and Eq.(3.2.18.) one obtain

$$= (-1)^{\mathbf{n}}(\mathbf{n}$$

$$^*\Theta_1$$

$$\mathbf{n}$$

$$M_k(n,p) = e^k\int_k^{+\infty}\left[\frac{x\left[^{p-1}(x-1)\ldots\right.}{(p-1}\right.$$

From Eq.(3.2.12.) and Eq.(3.2.21) one obtain

$$\int_0^{+\infty}\left[\frac{(u+k)^{p-1}[(u+k-1)\times\ldots\times u\times\ldots}{(p-1)!}\right.$$

$$= p\cdot\Theta_2(n,p),$$

$$\Theta_2(n,p)\in\mathbb{Z},$$

$$k\in\mathbb{N}.$$

Using transfer, from Eq.(3.2.25.) one obtain $\forall k\,(k\in\mathbb{N})$:

$$^*M_k(\mathbf{n},\mathbf{p}) = e^k\cdot{}^*\left(\int_k^{+\infty}\left[\frac{x\left[\mathbf{p}^{-1}(x-1)\ldots(x-\mathbf{n})\right]^{\mathbf{p}}e^{-x}}{(\mathbf{p}-1)!}\right]dx\right) =$$

$$= {}^*\left(\int_0^{+\infty}\left[\frac{(u+k)^{\mathbf{p}-1}[(u+k-1)\cdot\ldots\cdot u\cdot\ldots\cdot(u+k-\mathbf{n})]^{\mathbf{p}}e^{-u}}{(p-1)!}\right]du\right) = \quad (3.2.26)$$

$$= \mathbf{p}\times{}^*\Theta_2(\mathbf{n},\mathbf{p}),$$

$$^*\Theta_2(\mathbf{n},\mathbf{p})\in{}^*\mathbb{Z}_\infty,$$

$$k = 1,2,\ldots,k\in\mathbb{N},$$

$$\mathbf{n},\mathbf{p}\in{}^*\mathbb{N}_\infty.$$

Using transfer, from inequality (3.2.23.) one obtain $\forall k\,(k \in \mathbb{N})$:

$$ {}^*\varepsilon_k\left(\mathbf{n},\mathbf{p}\right) \leq {}^*g_k\left(\mathbf{n}\right)\frac{\left[{}^*a\left(\mathbf{n}\right)\right]^{\mathbf{p}-1}}{(\mathbf{p}-1)!} \times {}^*\left(\int\limits_0^k dx\right) \leq \frac{\mathbf{n}\cdot g_k\left(\mathbf{n}\right)\cdot\left[a\left(\mathbf{n}\right)\right]^{\mathbf{p}-1}}{(\mathbf{p}-1)!}, \tag{3.2.27} $$

$$ k = 1, 2, ..., k \in \mathbb{N}, $$

$$ \mathbf{n}, \mathbf{p} \in {}^*\mathbb{N}_\infty. $$

By using transfer again, from Eq.(3.2.15.) one obtain $\forall k\,(k \in \mathbb{N})$:

$$ {}^*\left(e^k\right) = \left({}^*e\right)^k = \frac{{}^*M_k\left(\mathbf{n},\mathbf{p}\right) + {}^*\varepsilon_k\left(\mathbf{n},\mathbf{p}\right)}{{}^*M_0\left(\mathbf{n},\mathbf{p}\right)}, $$

$$ k = 1, 2, ..., \tag{3.2.28} $$

$$ k \in \mathbb{N}, $$

$$ \mathbf{n}, \mathbf{p} \in {}^*\mathbb{N}_\infty. $$

(Part II) By using Eq.(3.2.28.) one obtain

$$ \left[{}^*\left(e^k\right)\right]^\# = \left[\left({}^*e\right)^\#\right]^k = \frac{\left[{}^*M_k\left(\mathbf{n},\mathbf{p}\right)\right]^\# + \left[{}^*\varepsilon\right.}{\left[{}^*M_0\left(\mathbf{n},\mathbf{p}\right)\right]^\#} $$

$$ k = 1, 2, ..., $$

$$ k \in \mathbb{N}, $$

$$ \mathbf{n}, \mathbf{p} \in {}^*\mathbb{N}_\infty. $$

By using Eq.(3.2.24.) one obtain

$$ \left[{}^*M_0\left(\mathbf{n},\mathbf{p}\right)\right]^\# = \left[(-1)^{\mathbf{n}}\right]^\#\left[(\mathbf{n}!)^{\mathbf{p}}\right]^\# + \mathbf{p}^\# \times \left[{}^*\Theta_1\left(\mathbf{n},\mathbf{p}\right)\right. $$

$$ \left[{}^*\Theta_1\left(\mathbf{n},\mathbf{p}\right)\right]^\# \in {}^*\mathbb{Z}_{\infty,\mathbf{d}}, $$

$$ \mathbf{n}, \mathbf{p} \in {}^*\mathbb{N}_\infty. $$

By using Eq.(3.2.26.) one obtain

$$ \left[{}^*M_k\left(\mathbf{n},\mathbf{p}\right)\right]^\# == \mathbf{p}^\# \times \left[\,{}^*\Theta_2\left(\mathbf{n},\mathbf{p}\right)\right]^\#, $$

$$ \left[{}^*\Theta_2\left(\mathbf{n},\mathbf{p}\right)\right]^\# \in {}^*\mathbb{Z}_{\infty,\mathbf{d}}, \tag{3.2.31} $$

$$ k = 1, 2, ..., k \in \mathbb{N}, $$

$$ \mathbf{n}, \mathbf{p} \in {}^*\mathbb{N}_\infty. $$

By using inequality (3.2.27) one obtain

$$\left[{}^{*}\varepsilon_{k}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}\leq\frac{\mathbf{n}^{\#}\cdot\left[g_{k}\left(\mathbf{n}\right)\right]^{\#}\cdot\left[\left[a\left(\mathbf{n}\right)\right]^{\mathbf{p}-1}\right]^{\#}}{\left[\left(\mathbf{p}-1\right)!\right]^{\#}},$$

$$k=1,2,...,k\in\mathbb{N}, \tag{3.2.32}$$

$$\mathbf{n},\mathbf{p}\in{}^{*}\mathbb{N}_{\infty}.$$

$$\Im_{0}^{\#}+\left[\overline{\overline{\#Ext\text{-}\sum_{n\in\mathbb{N}\backslash\{0\}}\Im_{n}^{\#}\times\left({}^{*}e^{n}\right)^{\#}}}\right|\Im^{\#}$$

Substitution Eq.(3.2.28) into Eq.(3.2.10) gives

$$\Im_{0}^{\#}+\left[\overline{\overline{\#Ext\text{-}\sum_{k=1}^{\infty}\Im_{k}^{\#}\times\frac{\left[{}^{*}M_{k}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}+\ \left[{}^{*}\varepsilon_{k}\left(\right.\right.}{\left[{}^{*}M_{0}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}}}}\right.$$

$$=\Im^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}},.$$

Multiplying Eq.(3.2.33) by number $\left[{}^{*}M_{0}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}\in{}^{*}\mathbb{Z}_{\mathbf{d}}$ one obtain

$$\Im_{0}^{\#}\times\ \left[{}^{*}M_{0}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}+$$

$$\left[\overline{\overline{\#Ext\text{-}\sum_{k=1}^{\infty}\ \left\{\Im_{k}^{\#}\times\left[{}^{*}M_{k}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}+\Im_{k}^{\#}\times\left[{}^{*}\varepsilon_{k}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}\right\}}}\right. \tag{3.2.34}$$

$$\left|\Im^{\#}\times\left[{}^{*}M_{0}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}\times\left({}^{*}c\right)^{\#}\right]_{\varepsilon}=$$

$$=\Im^{\#}\times\ \left[{}^{*}M_{0}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}.$$

By using inequality (3.2.32) for we will choose prime hyper number $\mathbf{p}\in{}^{*}\mathbb{N}_{\infty}$ given ε such that:

$$\left[\overline{\overline{\#Ext\text{-}\sum_{k=1}^{\infty}\ \Im_{k}^{\#}\times\left[{}^{*}\varepsilon_{k}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}}}\right|\Im^{\#}\times\left({}^{*}c\right)^{\#}\right]_{\varepsilon}\in \tag{3.2.35}$$

$$\in\Im^{\#}\times\ \left[{}^{*}M_{0}\left(\mathbf{n},\mathbf{p}\right)\right]^{\#}\times\varepsilon\times\varepsilon_{\mathbf{d}}.$$

Hence from Eq.(3.2.34) and (3.2.35) one obtain

$$\Im_0^{\#} \times \ [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} +$$

$$+ \left[\overline{\overline{\#Ext\text{-}\sum_{k=1}^{\infty} \ \Im_k^{\#} \times [{}^*M_k(\mathbf{n},\mathbf{p})]^{\#}}} \,\Big|\, \Im^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times ({}^*c)^{\#} \right]_{\varepsilon} = \tag{3.2.36}$$

$$= \Im^{\#} \times \ [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}.$$

We will choose prime hyper number $\mathbf{p} \in {}^*\mathbb{N}_{\infty}$ such that

$$\mathbf{p}^{\#} > \max\left(\Im^{\#}, \left|\Im_0^{\#}\right|, \mathbf{n}^{\#}.\right) \tag{3.2.37}$$

Hence by using Eq.(3.2.20) one obtain: $[{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \nmid \mathbf{p}^{\#}$

and consequently $[{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \neq 0^{\#}$.

And by using (3.2.20),(3.2.28) one obtain: $[{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \Im_0^{\#} \nmid \mathbf{p}^{\#}$.

By using Eq.(3.2.22) one obtain

$$[{}^*M_k(\mathbf{n},\mathbf{p})]^{\#} \mid \mathbf{p}^{\#}, \tag{3.2.40}$$

$$k = 1, 2, ...$$

By using Eq.(3.2.36) one obtain

$$\Xi(\mathbf{n},\mathbf{p},\boldsymbol{\varepsilon}) = \mathbf{a}.\mathbf{p}.\left\{[\Xi(\mathbf{n},\mathbf{p})]_{\varepsilon}\right\} =$$

$$\mathbf{ab}.\mathbf{p}.\left\{ \left[\overline{\overline{\#Ext\text{-}\sum_{k=1}^{\infty} \ \Im_k^{\#} \times [{}^*M_k(\mathbf{n},\mathbf{p})]^{\#}}} \,\Big|\, \Im^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times ({}^*c)^{\#} \right]_{\varepsilon} \right\} \tag{3.2.41}$$

$$= \Im^{\#} \times \ [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon^{\#} \times \varepsilon_{\mathbf{d}}.$$

$$\Xi(\mathbf{n},\mathbf{p}) = \overline{\overline{\#Ext\text{-}\sum_{k=1}^{\infty} \ \Im_k^{\#} \times [{}^*M_k(\mathbf{n},\mathbf{p})]^{\#}}}$$

It is easy to see that Wattenberg hypernatural number $\Xi(\mathbf{n},\mathbf{p})$ has tipe 1.Hence Wattenberg hypernatural number $\Xi(\mathbf{n},\mathbf{p})$ has

represantation:

$$\Xi(\mathbf{n},\mathbf{p}) = \mathbf{p}^{\#}\times\mathbf{m} + \Im^{\#} \times \ [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}, \qquad (3.$$

$$\mathbf{m} \in {}^*\mathbb{Z}_{\mathbf{d}}.$$

By using (3.2.42) one obtain represantation $$\Xi(\mathbf{n},\mathbf{p},\boldsymbol{\varepsilon}) = [\Xi(\mathbf{n},\mathbf{p})]_{\varepsilon} = \mathbf{p}^{\#}\times\mathbf{m}+\Im^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p}$$

$$\mathbf{m}\in{}^{*}\mathbb{Z}_{\mathbf{d}}.$$

Substitution Eq.(3.2.43) into Eq.(3.2.36) gives

$$\left\{\Im_0^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}+\mathbf{p}^{\#}\times\mathbf{m}\right\}+\Im^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}= \tag{3.2.45}$$

$$=\Im^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}.$$

By using (3.2.39)-(3.2.40) one obtain $$\left\{\Im_0^{\#}\times[{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}+\mathbf{p}^{\#}\times\mathbf{m}\right\}\nmid\mathbf{p}^{\#}$$

and consequently $\left\{\Im_0^{\#}\times[{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}+\mathbf{p}^{\#}\times\mathbf{m}\right\}\neq 0^{\#}$.But on the other hand, for sufficiently infinite smoll $\varepsilon\in{}^{*}\mathbb{R}$ idempotent $\Im^{\#}\times[{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}$ does not absorbs Wattenberg hypernatural number $\left\{\Im_0^{\#}\times[{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}+\mathbf{p}^{\#}\times\mathbf{m}\right\}$ and consequently:

$$\left\{\Im_0^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}+\mathbf{p}^{\#}\times\mathbf{m}\right\}+\Im^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}\neq \tag{3.2.47}$$

$$\neq\Im^{\#}\times\ [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}.$$

Thus for sufficiently infinite smoll $\varepsilon\in{}^{*}\mathbb{R}$ inequality (3.2.47) in a contradiction with Eq.(3.2.45).This contradiction proves that e is not w-transcendental. Hence e is #-transcendental.

Proof II. (Part I)

To prove e is #-transcendental we must show it is not w-transcendental, i.e., there is no real analytic function $g_{\mathbb{Q}}(x)=\sum_{n=0}^{\infty}b_n x^n, e\leq|x|\leq r$ with

rational coefficients $b_0,b_1,...,b_n,...\in\mathbb{Q}$ such that $$\sum_{n=0}^{\infty}b_n e^n=0.$$

$$b_0=\frac{k_0}{m_0}, b_n=\frac{k_n}{m_n}.$$

1. Assume that $b_0,b_1,...,b_n,...\in\mathbb{Q}, b_0\neq 0$.

Let $f(x):\mathbb{R}\to\mathbb{R}$ be a polynomial of degree $m\in\mathbb{N}$. Then (repeated) integrations by

parts gives

$$\int_0^k f(x)\,e^{-x}dx = -f(x)\,e^{-x}\Big|_0^k + \int_0^k f'(x)\,e^{-x}dx = \qquad (3.2.49)$$

$$= -\left(f(x)+f'(x)+...+f^{(m)}(x)\right)e^{-x}\Big|_0^k$$

Multiply by $b_k e^k \in {}^*\mathbb{Z}, k = 0,1,2,...,n \in \mathbb{N}$ and add up: Then $\sum_{k=0}^{n} b_k e^k \int_0$

$$= \left(f(0)+f'(0)+...+f^{(m)}(0)\right)\sum_{k=0}^{n} b$$

$$\sum_{k=0}^{n}\left({}^*b_k\left({}^*e^k\right)\right) \times {}^*\left(\int_0^k f(x)\,e^{-x}dx\right)$$

Using transfer from Eq.(3.2.50) one obtain $-\left({}^*f(0) + {}^*f'(0) + ... + {}^*f^{(\mathbf{m})}(0)\right)\sum_{k=0}^{n}$

$$= -\sum_{k=0}^{n}\left({}^*b_k\right)\left({}^*f(k) + {}^*f'(k) + ... + {}^*f\right.$$

$$\mathbf{m} \in {}^*\mathbb{N}_\infty.$$

Hence

$$\sum_{k=0}^{n}\left({}^*b_k\left({}^*e^k\right)\right) = \sum_{k=0}^{n}\left({}^*b_k\right)\left(\frac{\Delta_k}{\Delta_0} - \frac{\gamma_k}{\Delta_0}\right),$$

$$\gamma_k = \left({}^*b_k\left({}^*e^k\right)\right) \times {}^*\left(\int_0^k f(x)\,e^{-x}dx\right), \qquad (3.2.52)$$

$$\Delta_0 = \left({}^*f(0) + {}^*f'(0) + ... + {}^*f^{(\mathbf{m})}(0)\right),$$

$$\Delta_k = \left({}^*f(k) + {}^*f'(k) + ... + {}^*f^{(\mathbf{m})}(k)\right).$$

From Eq.(3.2.52) one obtain

$$\sum_{k=0}^{n} \left({}^{*}b_k\right)^{\#} \times \left({}^{*}e^{k}\right)^{\#} = \sum_{k=0}^{n} \left({}^{*}b_k\right)^{\#} \times \left(\frac{\Delta_k^{\#}}{\Delta_0^{\#}} - \frac{\gamma_k^{\#}}{\Delta_0^{\#}} \right),$$

$$\gamma_k^{\#} = \left(\left({}^{*}b_k \left({}^{*}e^{k}\right)\right) \times {}^{*}\left(\int_0^k f(x)\, e^{-x} dx \right) \right)^{\#}, \tag{3.2.53}$$

$$\Delta_0^{\#} = \left({}^{*}f(0) + {}^{*}f'(0) + \ldots + {}^{*}f^{(\mathbf{m})}(0)\right)^{\#},$$

$$\Delta_k^{\#} = \left({}^{*}f(k) + {}^{*}f'(k) + \ldots + {}^{*}f^{(\mathbf{m})}(k)\right)^{\#}.$$

2. We will choose $f(x)$ of the form
$$f(x) = \frac{1}{(P-1)!} x^{P-1} \cdot (x-1)^{P} \cdot (x-2)^{P} \cdot \ldots \cdot (x-n)$$
where $P \in \mathbb{N}$ is a prime number. Note that for $0 \leq x \leq n \in \mathbb{N}$

we have
$$|f(x)| \leq \frac{n^{(n+1)\cdot P}}{(P-1)!} = \frac{[A(n)]^{P}}{(P-1)!}, \tag{3.2.55}$$
$$A(n) = n^{n+1}.$$

Using transfer from (3.2.54)-(3.2.55) one obtain

$$^{*}(\ f(x)) = \frac{1}{(\mathbf{P}-1)!} x^{\mathbf{P}-1} \cdot (x-1)^{\mathbf{P}} \cdot (x-2)^{\mathbf{P}} \cdot \ldots \cdot (x-\mathbf{n})^{\mathbf{P}},$$

$$|^{*}(\ f(x))| \leq \frac{\mathbf{n}^{(\mathbf{n}+1)\cdot\mathbf{P}}}{(\mathbf{P}-1)!} = \frac{[A(\mathbf{n})]^{\mathbf{P}}}{(\mathbf{P}-1)!}, A(\mathbf{n}) = \mathbf{n}^{\mathbf{n}+1} \tag{3.2.56}$$

$$\mathbf{P}, \mathbf{n} \in {}^{*}\mathbb{N}_{\infty}.$$

Hence
$$\overline{\overline{\#\text{-}\sum_{k=0}^{\infty} \left| \left(\left({}^{*}b_k\right)^{\#} \times \left({}^{*}e^{k}\right)^{\#} \right) \times \left[{}^{*}\left(\int_0^k f(x)\, e^{-x} dx \right) \right]^{\#} \right|}} \leq$$
$$\leq \left(\overline{\overline{\#\text{-}\sum_{k=0}^{\infty} \left| \left({}^{*}b_k\right)^{\#} \right| \times \left({}^{*}e^{k}\right)^{\#}}} \right) \times \frac{\left([A(\mathbf{n})]^{\mathbf{P}} \right)^{\#}}{[(\mathbf{P}-1)!]^{\#}} \leq \tag{3.2.57}$$
$$\leq \frac{\Delta_{\mathbf{d}} \times \left([A(\mathbf{n})]^{\mathbf{P}} \right)^{\#}}{[(\mathbf{P}-1)!]^{\#}},$$

It is easy to see that for $\mathbf{P} \in {}^{*}\mathbb{N}_{\infty}$. large enough, this is less than $\epsilon^{\#}$ for a given $\epsilon \approx 0, \epsilon \in {}^{*}\mathbb{R}$

$$\overline{\overline{\#\text{-}\sum_{k=0}^{\infty} \left({}^{*}b_k\right)^{\#} \times \left({}^{*}e^{k}\right)^{\#}}} = \left({}^{*}b_0\right)$$

Thus for $\mathbf{P} \in {}^{*}\mathbb{N}_{\infty}$ large enough by using Eq.(3.2.53) one obtain

$$\overline{\overline{\# - \sum_{k=0}^{\infty} \Big|\gamma}}$$

$$\Delta_0^{\#} = \left({}^{*}f\left(0\right) + {}^{*}f\right.$$

$$\Delta_k^{\#} = \left({}^{*}f\left(k\right) + {}^{*}f\right.$$

Thus

$$\left[\overline{\overline{\#\text{-}\sum_{k=0}^{\infty} \left({}^{*}b_k\right)^{\#} \times \left({}^{*}e^{k}\right)^{\#}}}\right]_{\varepsilon} = \left({}^{*}b_0\right)^{\#} + \left[\overline{\overline{\#\text{-}\sum_{k=1}^{\infty} \left({}^{*}b_k\right)^{\#} \times \left(\frac{\Delta_k^{\#}}{\Delta_0^{\#}} - \frac{\gamma_k^{\#}}{\Delta_0^{\#}}\right)}}\right]_{\varepsilon}, \qquad (3.2.59)$$

$$\overline{\overline{\# - \sum_{k=0}^{\infty} \left|\gamma_k^{\#}\right|}} < \epsilon^{\#}, \epsilon \approx 0,$$

By using **Theorem 1.** and Eq.(3.2.48) one obtain

$$\overline{\overline{\#\text{-}\sum_{k=0}^{\infty} \left({}^{*}b_k\right)^{\#} \times \left({}^{*}e^{k}\right)^{\#}}} = \varepsilon_{\mathbf{d}}.$$

By using **Theorem 1.3.4** and Eq.(3.2.60) one obtain

$$\left[\overline{\overline{\#\text{-}\sum_{k=0}^{\infty} \left({}^{*}b_k\right)^{\#} \times \left({}^{*}e^{k}\right)^{\#}}}\right]_{\varepsilon} = [\varepsilon$$

$$[\varepsilon_{\mathbf{d}}]_{\varepsilon} = \left({}^{*}b_0\right)^{\#} + \left[\overline{\overline{\#\text{-}\sum_{k=1}^{\infty} \left({}^{*}b_k\right)^{\#} \times \Big(\frac{\Delta}{\Delta}}}\right.$$

By using Eq.(3.2.59) and Eq.(3.2.60) one obtain

$$\overline{\overline{\# - \sum_{k=0}^{\infty} \left|\gamma_k^{\#}\right|}} < \epsilon^{\#}\left(\varepsilon\right),$$

$$\epsilon, \varepsilon \approx 0.$$

Hence

$$\Delta_0^{\#}\times[\varepsilon_{\mathbf{d}}]_\varepsilon = \Delta_0^{\#}\times({}^*b_0)^{\#} + \left[\overline{\overline{\#\text{-}\sum_{k=1}^{\infty}({}^*b_k)^{\#}\times\left(\Delta_k^{\#}-\gamma_k^{\#}\right)}}\,\Big|\Delta_0^{\#}\times({}^*c)^{\#}\right]_\varepsilon, \qquad (3.2.63)$$

$$c\in\mathbb{R}.$$

Multiplying Eq.(3.2.63) by number $\Im^{\#}$, where

$\Im=(m_0, m_0\times m_1, ..., m_0\times m_1\times ...\times m_n, ...)$ gives

$$\Im^{\#}\times\Delta_0^{\#}\times\varepsilon^{\#}$$

$$=\Im^{\#}\times\Delta_0^{\#}\times({}^*b_0)^{\#}+\Im^{\#}\times\left[\overline{\overline{\#\text{-}\sum_{k=1}^{n}({}^*b_k}}\right.$$

thus

$$\Im^{\#}\times\Delta_0^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}=$$

$$=\Im^{\#}\times\Delta_0^{\#}\times({}^*b_0)^{\#}+\left[\overline{\overline{\#\text{-}\sum_{k=1}^{n}\left(\Im_k^{\#}\times\Delta_k^{\#}-\Im^{\#}\gamma_k^{\#}\right)}}\,\Big|\Im^{\#}\times\Delta_0^{\#}\times({}^*c)^{\#}\right]_\varepsilon=$$

$$=\Im^{\#}\times\Delta_0^{\#}\times({}^*b_0)^{\#}+\left[\overline{\overline{\#\text{-}\sum_{k=1}^{n}\left(\Im_k^{\#}\times\Delta_k^{\#}-\Im^{\#}\times\gamma_k^{\#}\right)}}\,\Big|\Im^{\#}\times\Delta_0^{\#}\times({}^*c)^{\#}\right]_\varepsilon, \qquad (3.2.65)$$

$$\Im^{\#}\times\left(\overline{\overline{\#-\sum_{k=0}^{\infty}\left|\gamma_k^{\#}\right|}}\right)<\epsilon^{\#}(\varepsilon)<\varepsilon^{\#}, \varepsilon\approx 0,$$

$$\Im_0^{\#}=\Im^{\#}\times({}^*b_0)^{\#}, \Im_k^{\#}=\Im^{\#}\times({}^*b_k)^{\#}, k=1,2,...$$

3. If
$$h(x)=\frac{g(x)(x-a)^{\mathbf{P}}}{\mathbf{P}!} \qquad (3.2.66)$$

$$\mathbf{P}\in{}^*\mathbb{N}_\infty,$$

where $g(x)$ is any hyper polynomial with hyper integer coefficients and $a\in{}^*\mathbb{Z}$ then the derivatives $h^{(j)}(a)=0$ for $0\leq j<\mathbf{P}$ and in general $h^{(j)}(a)\in{}^*\mathbb{Z}$ for all $j\geq 0$. Since $f(x)/\mathbf{P}$ has this form with $a\in\{1,2,...,\mathbf{n}\}$ it follows that $f^{(j)}(k)$ is an integer and is divisible by $\mathbf{P}$, for all $j\geq 0$ and for $k\in\{1,2,...,\mathbf{n}\}$. Thus $\mathbf{P}$ divides all terms on the **RHS** of Eq.(3.2.65) having $k\neq 0$.

4. It remains to consider the terms with $k=0$.Note that ${}^*f(x)$

has the form $$^*f(x)=\sum_{\mathbf{j}=\mathbf{P}-1}^{\mathbf{m}}\frac{c_{\mathbf{j}}x^{\mathbf{j}}}{(\mathbf{P}-1)!} \quad (3.2.$$

where $c_{\mathbf{P}-1}=(\pm\,\mathbf{n}!)^{\mathbf{P}}$ and $c_{\mathbf{j}}\in{}^*\mathbb{Z}$ for all $\mathbf{j}\in{}^*\mathbb{N}$.Then $^*f^{\,(\mathbf{j})}(0)=0$ for $\mathbf{j}<\mathbf{P}-1$, $^*f^{\,(\mathbf{P}-1)}(0)=c_{\mathbf{P}-1}$ and $^*f^{\,(\mathbf{j})}(0)=c_{\mathbf{j}}\cdot\mathbf{j}!/(\mathbf{P}-1)!$ for $\mathbf{j}\geq\mathbf{P}$ so $\mathbf{P}$ divides $^*f^{\,(\mathbf{j})}(0)$ if $\mathbf{j}\neq\mathbf{P}-1$.

5. The only term remaining on the **RHS** of Eq.(3.2.65) is $$\Im^{\#}\times\Delta_0^{\#}\times({}^*b_0)^{\#}\times\left({}^*f^{\,(\mathbf{P}-1)}\right.$$

This term is not divisible by $\mathbf{P}^{\#}\in{}^*\mathbb{N}_{\infty,\mathbf{d}}$ if $\mathbf{P}\in{}^*\mathbb{N}_{\infty}$ is prime with $\mathbf{P}>|{}^*b_0|\times\mathbf{n}$.Thus, we may choose $\mathbf{P}$ so that $\Im^{\#}\times\left(\overline{\overline{\#-\sum_{k=0}^{\infty}\left|\gamma_k^{\#}\right|}}\right)<\varepsilon^{\#}$ and so that in the **RHS** of Eq.(3.2.65), $\mathbf{P}$ divides every term

$$\Im^{\#}\times\Delta_0^{\#}\times\left({}^{\#}b_k\right)\times\left({}^*f^{\,(\mathbf{j})}(k)\right)^{\#},\mathbf{j}\in{}^*\mathbb{N} \tag{3.2.69}$$

except for $\Im^{\#}\times\Delta_0^{\#}\times\left({}^{\#}b_0\right)\times\left({}^*f^{\,(\mathbf{P}-1)}(0)\right)^{\#}$. Therefore the **RHS** has representation $\Gamma^{\#}+\Im^{\#}\times\Delta_0^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}$ such that $\Gamma\in{}^*\mathbb{R},\ \Gamma\geq 1$.
Thus one obtain

$$\Im^{\#}\times\Delta_0^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}=\Gamma^{\#}+\Im^{\#}\times\Delta_0^{\#}\times\varepsilon^{\#}\times\varepsilon_{\mathbf{d}}$$

This is a contradiction.This contradiction proves that e is not w-transcendental. Hence e is $\#$-transcendental.

38 III.3.Nonstandard generalization of the Lindeman Theorem.

Theorem 3.3.1.(Nonstandard Lindeman Theorem) The number $^*e$

cannot satisfy an equation of the next form: $$a_1\cdot({}^*e)^{\alpha_1}+a_2\cdot({}^*e)^{\alpha_2}+...+a_N\cdot({}^*e)^{\alpha_N}$$

in which at least one coefficient $a_n, n=1,2,...,N\in{}^*\mathbb{N}_{\infty}$ is different from zero, no two exponents $\alpha_n, n=1,2,...,N\in{}^*\mathbb{N}_{\infty}$ are equal, and all numbers $\alpha_n, n=1,2,...,N\in{}^*\mathbb{N}_{\infty}$ are hyperalgebraic.
Proposition 3.3.1. Let $\rho_1,\rho_2,...,\rho_m,\ m\in{}^*\mathbb{N}_{\infty}$ be the roots of the hyperpolynomial equation $a\cdot z^m+b\cdot z^{m-1}+c\cdot z^{m-2}+...=0$ with integral coefficients $a,b,c,...\in{}^*\mathbb{Z}$. Then any symmetric hyperpolynomial in the

quantities $a \cdot \rho_1, a \cdot \rho_2, ..., a \cdot \rho_m$ with integral coefficients, is an hyperinteger.

Proposition 3.3.2. Suppose given a hyperpolynomial in $m \in {}^*\mathbb{N}_\infty$ variables α_{i_1},
in $n \in {}^*\mathbb{N}_\infty$ variables $\beta_{i_2}, ...$,and in $k \in {}^*\mathbb{N}_\infty$ variables $\sigma_{i_l}, l \in {}^*\mathbb{N}_\infty$ which is symmetric in the α's in the β's,...,and in the σ's, and which has hyperrational coefficients. If the α's are chosen to be all the roots of a hyperpolynomial equation with rational coefficients, and similarly for the β's, , and for the σ's
then the value of the polynomial is hyperrational.

Definition 3.3.1.A hyperpolynomial is said to be irreducible over the rationale
if it cannot be factored into hyperpolynomials of lower degree with hyperrational
coefficients.

Definition 3.3.2. If α_1 is a root of an irreducible hyperpolynomial equation with
hyperrational coefficients, whose other roots are $\alpha_2, \alpha_3, , ..., \alpha_n, n \in^* \mathbb{N}$ then the
hyperalgebraic numbers $\alpha_1, \alpha_2, \alpha_3, , ..., \alpha_n$ are said to be the *conjugates* of α_1.

Proposition 3.3.3. Any hyperpolynomial with hyperrational coefficients can be
factored into irreducible polynomials with hyperrational coefficients.

Proposition 3.3.4. Over the field ${}^*\mathbb{Q}$, an hyperalgebraic number is a root of a
unique irreducible hyperpolynomial with hyperrational coefficients and leading
coefficient unity. Such an equation has no multiple roots.

Proposition 3.3.5. The Van der Monde determinant $\det \left| (\rho_k)^{i-1} \right|$ vanishes only
if two or more of the ρ's are equal.

39 III.4. The numbers e and π are analytically independent.

First of all, recall that is an entire function,in 2 variables, with
coefficients in field $\mathbb{Q}$,is a function $f(z_1, z_2)$ which is analytic in $G \subsetneqq \mathbb{C} \times \mathbb{C}$

$$f(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} z_1^i z_2^j, \tag{3.4.1}$$

$$c_{i,j} \in \mathbb{Q}.$$

Definition 3.3.1. Two complex numbers $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ are said to be analytically dependent if there is a nonzero entire function $f(z_1, z_2)$ in 2 variables, with hyperinteger coefficients $c_{i,j} \in \mathbb{Q}$, such that $f(\alpha, \beta) = 0$. Otherwise, $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ are said to be analytically independent.

40 Appendix A. Hyper algebraic numbers.

41 1.Definitions of symmetric polynomials and symmetric functionals.

Consider a monic polynomial $P(z)$ in $z \in \mathbb{C}$ of degree $n \in \mathbb{N}$ $P(z) = 1 + a_1 z + a_2 z^2 + ... + a$

There exist n roots $z_1, \ldots, z_n$ of P and that one is expressed by the relation

$$P(z) = 1 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1} + a_n z^n =$$

$$= \left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right) \cdots \left(1 - \frac{z}{z_n}\right) =$$

$$z_1^{-1} \cdot z_2^{-1} \cdots z_n^{-1} (z_1 - z) \cdot (z_2 - z) \cdots (z_n - z) = \quad (A$$

$$= (-1)^n z_1^{-1} \cdot z_2^{-1} \cdots z_n^{-1} (z - z_1) \cdot (z - z_2) \cdots (z - z_n)$$

$$\hat{P}(z) = P(z) / (-1)^n z_1^{-1} \cdot z_2^{-1} \cdots z_n^{-1} =$$

$$= \hat{a}_n z^n + \hat{a}_{n-1} z^{n-1} + ... + \hat{a}_2 z^2 + \hat{a}_1 z + \hat{a}_0$$

Thus by comparison of the coefficients one finds

$$a_1 = -\sum_{1 \le i \le n} \frac{1}{z_i},$$

$$a_2 = \sum_{1 \le i < j \le n} \frac{1}{z_i z_j},$$

$$\cdots\cdots$$

$$a_m =$$

$$\cdots\cdots$$

$$a_{n-1} = (-1)^{n-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{n-1} \le n} \prod$$

$$a_n = (-1)^n \prod_{1 \le i \le n} \frac{1}{z_i}.$$

Definition. Let us defined n polynomials expressed by the relations

$$e_1(z_1, ..., z_n) = \sum_{1 \le i \le n} \frac{1}{z_i} = -a_1,$$

$$e_2(z_1, ..., z_n) = \sum_{1 \le i < j \le n} \frac{1}{z_i z_j} = a_2,$$

$$\cdots\cdots$$

$$e_m(z_1, ..., z_n) = \qquad (A.1.4)$$

$$\cdots\cdots$$

$$e_{n-1}(z_1, ..., z_n) = a_{n-1} =$$

$$e_n(z_1, ..., z_n) = a_n = (-1)^n \prod_{1 \le i \le n} \frac{1}{z_i}.$$

The polynomial $e_m(z_1, ..., z_n)$ is called the m-th symmetric polynomial.
It has the following property:

$$(A.1.5)$$

42 2.Nonstandard polynomials.

The set of natural,integer,rational, real, complex or any algebraic numbers

is denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{k}$ respectively, and their nonstandard extensions ${}^*\mathbb{N}, {}^*\mathbb{Z}, {}^*\mathbb{Q}, {}^*\mathbb{R}, {}^*\mathbb{C}, {}^*\mathbb{k}$.

Definition A.2.1.(**Nonstandard polynomials**) Nonstandard polynomial of

hyper degree $\mathbf{d} \in {}^*\mathbb{N}_\infty$ in x with coefficients in nonstandard field ${}^*\mathbb{k}$ is an

expression defined by internal hyper finite sum of the form $f(x) =$

Definition A.2.2. (**Algebraic hyper integers**) If $\alpha \in {}^*\mathbb{C}$ is a root of a monic nonstandard polynomial of hyper degree $\mathbf{d} \in {}^*\mathbb{N}_\infty$,namely a root of

a polynomial of the form $f(x) = \sum_{j=0}^{\mathbf{d}} a_j x^j = a_0 + a_1 x + \ldots + a_{\mathbf{d}-1} x^{\mathbf{d}-1}$

$\forall j \left[a_j \in {}^*\mathbb{Z} \right]$

and α is not the root of such a polynomial of hyper degree less then $\mathbf{d}$, then α is colled an *algebraic hyper integer* of hyper degree $\mathbf{d} \in {}^*\mathbb{N}_\infty$.

Definition A.2.3. (**Nonstandard algebraic numbers**) An nonstandard algebraic number α of hyper degree $\mathbf{d} \in {}^*\mathbb{N}_\infty$ is a root of a monic nonstandard polynomial of hyper degree $\mathbf{d} \in {}^*\mathbb{N}_\infty$, and not be the root of an the nonstandard polynomial of hyper degree $\mathbf{d}_1 \in {}^*\mathbb{N}_\infty$ less then $\mathbf{d}$.

Remark. We have to mach examles standard real numbers that are not *standard algebraic numbers*, such as $\ln 2$ and π.These are examples of *standard transcendental numbers*, which are not standard algebraic numbers

We will establish that every hyper finite extension of ${}^*\mathbb{Q}$

Definition A.2.4. (**Simple hyper finite extentions and nonstandard polynomials**) If $\alpha \in {}^*E$ an hyper finite extention field of a given nonstandard field ${}^*F$,then α is colled **hyper algebraic** over ${}^*F$ if $f(\alpha) = 0$ for some nonzero $f(x) \in F[x]$ .If α

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